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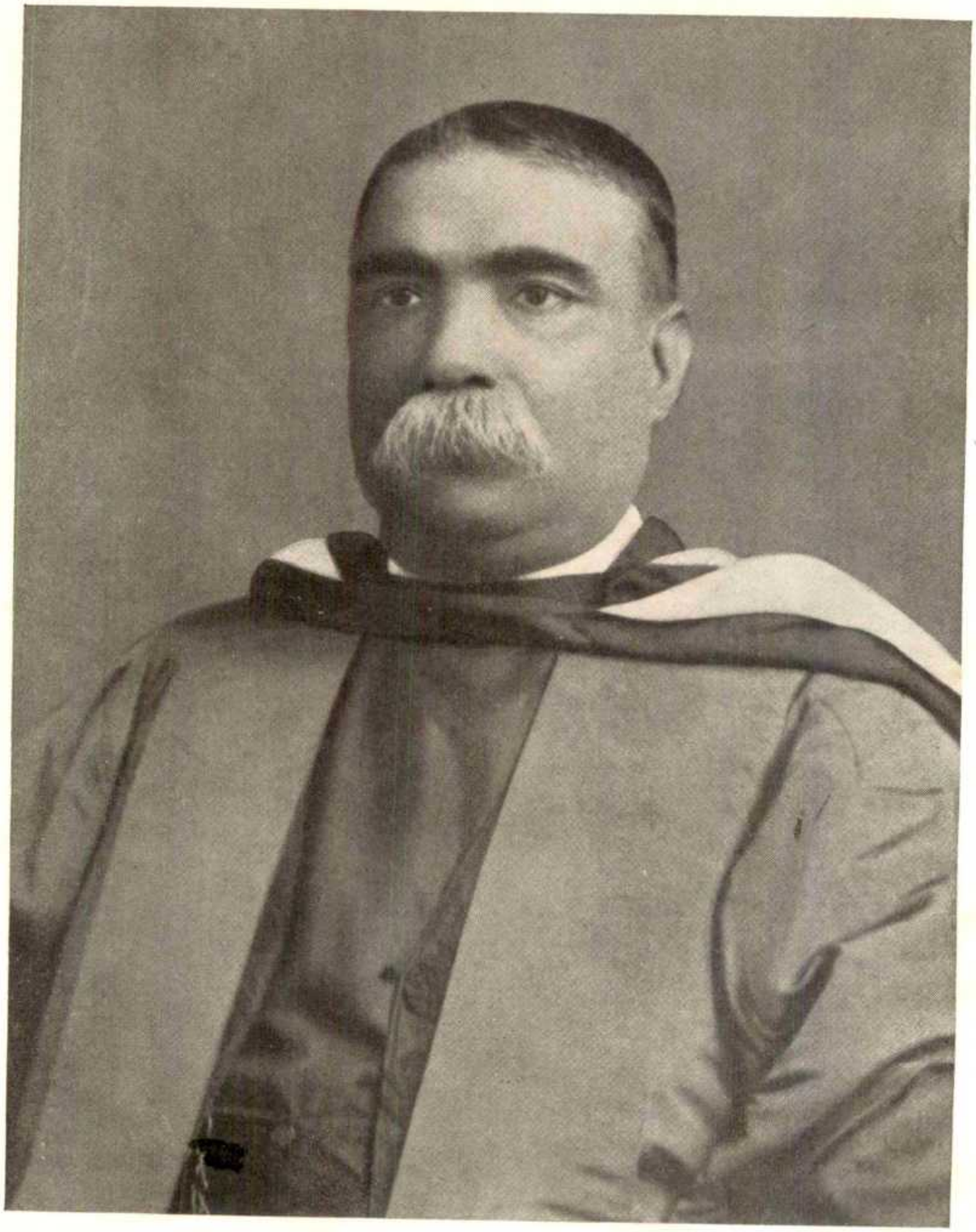

Sir Asutosh Mookerjee

(1864 – 1924)

Founder President of the Calcutta Mathematical Society

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SIR ASUTOSH MOOKERJEE

By

R. N. SEN, *Calcutta*

(*Received—October* 6, 1964)

I. LIFE SKETCH

Sir Asutosh Mookerjee was born in Calcutta on June 29, 1864. His father Dr. Gangaprasad Mookerjee was an eminent physician who had built up a good practice in his profession. In his boyhood Asutosh showed unmistakable signs of promise for the future and he grew up under the direct care and supervision of his father. In the formative period of life, he imbibed a keen desire for learning and inherited the strength of character from his parents. After finishing a preliminary course of study in a vernacular school, Asutosh was admitted in 1875 into the South Subarban School. In 1879 he matriculated at the age of fifteen, standing second in order of merit among the successful candidates, and took admission in the Presidency College, Calcutta. Early in his undergraduate course in this college he attracted the attention of his teachers and fellow students as a young prodigy, and his academic career was one of uniform brilliance. In 1884 he topped the list of successful candidates in the B. A. examination winning the Harischandra Prize, and continued his post-graduate study in the same college. The same year he joined the law classes in the City College, and won Tagore Law gold medal for three successive years, 1884-1886. In 1885 he took the M.A. degree in Mathematics securing first position in first class. In 1886 he took M.A. again in Natural Science and in the same year he was awarded Premchand Roychand Studentship in Mathematics and Science on the result of a competitive examination. In 1888 the took the B.L. degree and enrolled himself as a Vakil of the Calcutta High Court. In 1894 he was awarded the degree of Doctorate of Laws on a thesis. In 1897 he was elected as the Tagore Professor of Law and in 1908 the degree of Doctor of Science, Honoris Causa, was conferred on him.

In 1889 he became a Fellow of the Calcutta University and henceforth he remained intimately associated with the University all through his life. The first three Universities in India, those of Calcutta, Bombay and Madras, came into existence by an Act in 1857 and they were run entirely under the control of Government of India. In 1899 he entered Bengal Legislative Council as a representative of the Calcutta University and was re-elected there for two more terms, the last term from the Calcutta Corporation. In 1903 the Bengal Council sent him to the Imperial Legislative Council as its representative. In the same year the Government of India introduced a Bill in this Council which ultimately emerged out as the Indian Universities' Act of 1904 replacing the Act of 1857. By this Act, members from the public came to be associated with the administration of the Universities. But the Government wanted to retain control over the Universities, as most of these members and the Vice-Chancellors were nominated by the Government and there was no provision in the Act for statutory financial grant. However, the debate on the Bill in the Council which was presided over by the then Viceroy and Governor General of India, Lord Curzon, revealed Sir Asutosh's personality : independent, courageous, well-equipped in sound educational principles and constructive with mastery over details.

In 1904 Sir Asutosh was appointed a Judge of the Calcutta High Court ; the various law reports of the time proclaim his erudition, legal acumen and understanding. In 1906 he was invited by Lord Minto to take up the office of the Vice-Chancellor of the Calcutta University. Here began the towering career dedicated to the service to his *alma mater*. The work that he did in the University disclosed that he had in combination with his wide and deep intellectual interests a rare administrative ability and statesmanship. Inspite of many handicaps inherent in the Act of 1904 and against heavy odds, he carried the Senate with him to make plans, schemes and Regulations for stimulating and spreading education in the country. He was instrumental in establishing in 1909 the University Law College and in 1914 the University College of Science and Technology. His crowning achievement was the creation in 1917 of the Post-Graduate Departments of Teaching in Arts and Science in the University which provided opportunity and incentive for higher study and research. To achieve all these objectives he was able to raise generous and princely donations and to attract the most learned and talented persons in science, humanities and letters from all over India to run the Post-Graduate Departments. The University was transformed from being merely an affiliating and examining institution into an organization with the added responsibility of disseminating and unfolding of knowledge with the motto "advancement of learning." The creation of the Post-Graduate Departments, by which higher study and research were centralised, was a step of far reaching significance and a period of Renaissance followed. Indeed, Sir Asutosh was undoubtedly one of the makers of modern India.

Sir Asutosh remained Vice-Chancellor of the Calcutta University for four consecutive terms, from 1906 to 1914, and again he was appointed Vice-Chancellor in 1921 for a fresh term. As a matter of fact, from the time of his first association with the University as a Fellow till his death he gradually rose to become the dominating personality in the affairs of the University, occupying positions such as President of the Councils of Post-Graduate Teaching, of the various Boards of Studies etc. He raised the status of Indian Vernaculars, Indian History and Indian Philosophy. His domain of knowledge was vast and his memory astounding. He was a source of encouragement and inspiration to academic institutions and to individuals engaged in the pursuit of knowledge. In his time many secondary schools and undergraduate colleges sprang up in different parts of the province under private initiative and benevolence which became affiliated to the University. In short, he gave dynamic impetus to education. The then Government of Bengal under Lord Lytton could not reconcile with the educational policy pursued in the University by Sir Asutosh and there was constant rift between the University and the Government. As a creative visionary and a man of independent spirit, Sir Asutosh stood firmly against governmental interference, and maintained the freedom of the University even under most trying circumstances. In connection with this tussle, his words "freedom first, freedom second, freedom always" have since become well-known.

The annual convocation addresses which Sir Asutosh delivered as the Vice-Chancellor of the Calcutta University reveal his vision and judgment, his devotion and determination, his courage and independence. A few excerpts from these addressess may be quoted here to illustrate the ideals set by him. In a passage of his address of 1908 he said "that European knowledge should be brought home to our students through the medium of English—that western light should reach us through the western gates and not through lattice work in eastern windows." In his address of 1922 he said "To my mind the University is a great store house of learning, a great bureau of standards, a great workshop of knowledge, a great laboratory for the training as well of men of thought as men of action. The University is thus the instrument of the state for the conservation of knowledge, for the discovery of knowledge, for the distribution of knowledge, for the applications of knowledge, and above all, for the creation of knowledge-makers." In his address of 1923 he said "We stand unreservedly by the doctrine that if education is to be our policy as a nation, it must not be our politics; freedom is its very life-blood, the condition of its growth, the secret of its success......there stands forth unshaken the conviction that our insistent claim for the freedom of the University is a fight for the most sacred and impalpable of national privileges."

Sir Asutosh was a versatile genius, but his main interest lay in the field of Mathematics. During the ten years from 1880 to 1890, that is, during the years when he was preparing for and taking his B.A., M.A. and other examinations, he published nearly twenty original mathematical papers of high quality and merit. This is an accomplishment which has few parallels in the history of the mathematical world. These papers were published in the Messenger of Mathematics, the Quarterly Journal of Pure and Applied Mathematics and the Journal of the Asiatic Society of Bengal ; and he was elected a Fellow of the Royal Society of Edinburgh, of the Royal Astronomical Society, a member of the Royal Irish Academy, of the Mathematical Societies of London, Edinburgh, Paris, Palermo and New York. When he was only a first year student of the Presidency College, Asutosh published his first paper in which he gave an elegant new proof of the 25th proposition of the first book of Euclid. While still an undergraduate student, he published his second paper in which he gave some extensions of a theorem enunciated by Salmon. Whilst yet a student of the fifth year class he wrote his third paper 'A note on elliptic functions" which was of outstanding merit. In this paper he established a certain addition theorem in the theory of elliptic functions by a new method based on the properties of the ellipse. With regard to this paper Prof. Cayley observed that it was remarkable how a real result was obtained by consideration of an imaginary point. This paper has been referred to in Enneper's "Elliptische Funktionen". Immediately after taking his M.A. degree, Asutosh published a paper "On the differential equation of a trajectory." In this paper he took into consideration the solution given by the Italian mathematician Mainardi of the differential equation of oblique trajectory of confocal ellipses. This solution was so complicated, and inelegant that it was impossible for any one to trace the curve from it. Asutosh showed by an ingenious process that Mainardi's solution could be replaced by a pair of remarkably simple equations which admitted an interesting geometrical interpretation. These equations have been quoted by Prof. Forsyth in his book on Differential Equations in later editions. This paper on a particular trajectory led him to publish another paper on general trajectory which showed his power of generalisation and elegant expression. In this connection the following account may be mentioned : Asutosh read Mathematics with Prof. W. Booth in the Presidency College. Booth mentioned to Asutosh that Prof. M. Roberts in his lectures on differential equations at the University of Dublin used elliptic coordinates to solve Mainardi's problem. Asutosh had no opportunity to examine the solution, if any, arrived at by Roberts and it was believed that the solution had never been published. Booth's mention of it set Asutosh working and he soon found out the result by the help of elliptic coordinates. Again, during the years 1887-1889 Asutosh took up, in a series of three papers, the study of the differential equation of the general conic which was obtained by the French mathematician Monge. These papers roused a good deal of interest among a section of mathematicians. With regard to Monge's equation, Prof. Boole in his book on Differential Equations remarked that the geometrical interpretation of the equation had not so far been obtained. Later two geometric interpretations of the Mongian were proposed, one by Prof. Sylvester and the other by Lieut. Col. Cunningham. Asutosh in his papers critically examined these interpretations and showed that none of them could be accepted as a true interpretation as contemplated by Boole. Finally he gave his own geometrical interpretation of the Mongian which was accepted on all hands. This interpretation has been quoted by Edwards in his book on Differential Calculus. In a letter to Asutosh from Cambridge, dated the 14th September, 1887, Cayley remarked about his criticism of Sylvester's interpretation that "it is of course all perfectly right". Cunningham wrote "Professor Asutosh Mukhopadhyay has proposed a really excellent mode of geometric interpretation of differential equations in general, *viz.*, writing the equation in the form $F=0$, the geometric meaning of the symbol F considered as a magnitude (angle, length, area etc.) in any curve whatever (wherein F is of course not zero) is, if possible, to be formed ; then the geometric meaning of that equation obviously is that the quantity F vanishes right round every curve of the family represented by the differential equation. This is the most direct geometric interpretation yet proposed." (Nature, Vol. 38, pp. 318-19). A list of Sir Asutosh's papers and summaries of them are appended at the end. It may finally be mentioned that in 1893 he published a text book on Geometrical Conics, meant for the beginners of the subject. The book was highly reviewed and it ran into many editions.

Thus as a mathematical investigator, Sir Asutosh's keen insight and power of original thinking revealed themselves very early in his life. Truly Prof. Ganesh Prasad wrote "Sir Asutosh's contributions to mathematical knowledge were due to his unaided efforts while he was only a college student. After Bhaskara, he was the first Indian to enter into the field of mathematical researches, as distinguished from astronomical researches, and did much which was truly original." It can be said without the least hesitation that if Sir Asutosh had chosen to devote himself entirely to the study of Mathematics, he could have secured for himself a place in the front rank of world mathematicians. But fate ordained otherwise. In his convocation address of 1914 Sir Asutosh said in a pensive mood

"For years now, every hour, every minute I could spare from other unavoidable duties—foremost among them the duties of my judicial office—has been devoted by me to University work. Plans and schemes to heighten the efficiency of the University have been the subject of my day dreams, they have haunted me in the hours of nightly rest. To University concerns I have sacrificed all chances of study and research, possibly, to some extent, the interests of family and friends, and certainly, I regret to say, a good part of my health and vitality. Do not imagine, however, that I repine at the sacrifices made. I have had my reward in many ways."

Apart from his connection with the University of Calcutta, Sir Asutosh was associated with a number of scientific, humanistic and literary institutions. In 1907 he was elected President of the Asiatic Society of Bengal to which office he was repeatedly elected. In 1909 he became the Chairman of the Board of Trustees of the Indian Museum. In 1913 the Indian Science Congress Association came into existence and he was its first President. He brought into existence the Calcutta Mathematical Society for encouragement and promotion of mathematical researches and developed it under his fostering care as described below :

In 1908 Sir Asutosh founded the Calcutta Mathematical Society and guided, as President, the activities of the Society from its inception till his death. The inaugural meeting of the Society was held on the 6th September, 1908, at the Senate House of the Calcutta University at which 27 members were present and over which he presided. At the next two meetings, Constitution and Rules of the Society were adopted and the first Council was formed with Sir Asutosh Mookerjee as President, Sir Gurudas Banerjee, Prof. C. E. Cullis and Prof. G. S. De as Vice-Presidents, Rai Bahadur A. C. Bose as Treasurer and P. L. Ganguli as Secretary. The first publication of the Bulletin of the Society came out in 1909 and later, at the initiative of Sir Asutosh, the Calcuttta University Press undertook to print the Bulletin from its second volume. Society's meetings continued to be held at the Senate House until the meeting which was held at the Science College on the 8th October, 1915. All subsequent meetings of the Society were held in the Society's room at the Science College. During Sir Asutosh's life time 83 meetings of the Society (other than Council meetings) were held, the last meeting being held on the 1st March, 1924 ; and at *every one* of these meetings Sir Asutosh presided. The number of ordinary members of the Society (resident and non-resident) at the close of 1912 rose to 195, and by this time quite a good number of eminent foreign mathematicians were made honorary members of the Society. Not a few of the members of the Society, ordinary and honorary, presented mathematical books to the library of the Society, and monetary donations were received from benevolent persons from time to time. A large number of mathematical journals from all parts of the world were received on exchange basis with the Bulletin. In 1909 Prof. Ganesh Prasad, in 1913 Prof. A. R. Forsyth and in 1915 Prof. W. H. Young delivered addresses on their researches before the Society. Under Sir Asutosh's personal initiative, encouragement and inspiration, research workers, some of them very brilliant, began to gather round him whose contributions were published in the pages of the Bulletin. Papers were read at the Society for the first time by Profs. C. E. Cullis, Ganesh Prasad, S. D, Mookerjee, P. L. Ganguli and S. C. Bagchi in 1909, A. C. Bose and M. N. Roy in 1910, L. Narayan and J. C. Narayan in 1912, J. M. Bose in 1913, B. B. Datta in 1914, Sir C. V. Raman, F. R. S., N. L., M. N. Saha, F. R. S., N. K. Majumdar and S. K. Banerjee in 1915, S. N. Bose, F. R. S. and N. M. Basu in 1916, D. N. Mallik, N. R. Sen, S. M. Ganguli, H. P. Banerjee, S. C. Dhar and H. N. Datta in 1917, Sir B. N. Seal, S. C. Kar, B. N. Pal and P. C. Sen Gupta in 1918, S. K. Mitra, F. R. S., B. M. Sen, A. B. Batta and N. N. Sen in 1919, N. N. Ghosh and J. Ghosh in 1920, B. B. Roy, G. Bhar, O. Upadhyay, Hanumanta Rao and S. C. Mitra in 1922, M. Ghosh and B. B. Sen in 1923.

Sri Asutosh retired from the post of High Court Judge in 1923 and he died at Patna on May 25, 1924. His dead body was brought to Calcutta, the place of his activities, for cremation. Innumerable meetings were held to condole his death. Excerpts from the writings of two of the distinguished persons who paid their homages to the departed soul may be quoted here. Poet Rabindranath Tagore, N. L. wrote "Men are always rare in all countries through whom the aspiration of their people can hope to find its fulfilment, who have the thundering voice to say that what is needed shall be done ; Asutosh had the magic voice of assurance. He had the courage to dream because he had the power to fight and the confidence to win,—his will itself was the path to the goal," Sir Michael Sadler, who was the Chairman of the Calcutta University Commission (1917-19) of which Sir Asutosh was a member wrote "In Sir Asutosh Mookerjee India has lost one of her greatest men : the world one of its commanding personalities. He was mighty in battle. He could have ruled an empire. But he gave the best of his powers to Education because he believed that in Education rightly interpreted lies the secret of human welfare and the key to every empire's moral strength."

II. MATHEMATICAL PAPERS BY SIR ASUTOSH MOOKERJEE

1. Proof of Euclid I, 25,
   —Messenger of Mathematics, Vol. 10 (1880-81), pp. 122-123.
2. Extension of a theorem of Salmon's,
   —Messenger of Mathematics, Vol. 13 (1883-84), pp. 157-160.
3. A note on elliptic functions,
   —Quart. Jour. of Pure and Appld. Maths., Vol. 21 (1886), pp. 212-217.
4. On the differential equation of a trajectory,
   —Jour. Asiatic Soc. Beng., Vol. 56, Pt. II, No. 1 (1887), pp 117-120.
5. On Monge's differential equation to all conics,
   —Jour. Asiatic Soc. Beng., Vol. 56, Pt. II, No. 2 (1887), pp 134-145.
6. A memoir on plane analytical geometry,
   —Journ. Asiatic Soc. Beng., Vol. 56, Pt. II, No. 3 (1887), pp 288-349.
7. A general theorem on the differential equations of trajectories,
   —Journ. Asiatic Soc. Beng., Vol. 57, Pt. II, No. 1 (1888), pp 72-99.
8. On Poisson's integral,
   —Jour. Asiatic Soc. Beng., Vol. 57, Pt. II, No. 1 (1888), pp. 100-106.
9. On the differential equation of all parabolas,
   —Jour. Asiatic Soc. Beng., Vol. 57, Pt. II, No. 4 (1888), pp 316-332.
10. Remarks on Monge's differential equation to all conics,
   —Proc. Asiatic Soc. Beng., February 1888.
11. The geometric interpretation of Monge's differential equation to all conics,
   —Jour, Asiatic Soc. Beng., Vol. 58, Pt. II, No. 2 (1889), pp. 181-185,

12,13. Some applications of elliptic functions to problems of mean values, First and Second Papers.
—Jour. Asiatic Soc. Beng., Vol. 58, Pt. II, No. 2 (1889), pp. 199-213, 213-231.
14. On Clebsch's transformation of the hydrokinetic equations,
—Journ. Asiatic Soc Beng., Vol. 59, Pt. II No. 1 (1890), pp. 56-59.
15. Note on Stoke's theorem and hydrokinetic circulation,
—Jour. Asiatic Soc. Beng., Vol. 59, Pt. II, No. 1 (1890), pp. 59-61.
16. On a curve of aberrancy,
—Jour. Asiatic Soc. Beng. Vol. 59, Pt. II, No. 1 (1890), pp. 61-63.
17. Mathematical Notes (Questions and Solutions),
—Educational Times, Lond., (1890-92), Vols. 43 (pp. 125-151), 44 (pp. 144-182), 45 (pp. 146-168),

III. SUMMARIES OF SIR ASUTOSH MOOKERJEE'S MATHEMATICAL PAPERS

1. Proof of Euclid I, 25.

In this paper Mookerjee has given a new proof of the 25th proposition of the first book of Euclid (which states that if in any two triangles ABC and DEF, $AB=DE$, $AC=DF$ and $BC>EF$ then $\sphericalangle BAC> \sphericalangle EDF$) by the method of superposition, namely E is taken to coincide with B, EF to fall on BC and BD to be coterminous with AB. This new proof is short and it replaces a lengthy proof (that of Euc. I, 24 on which it used to depend). The author says that this direct proof was discovered by him nearly five years ago, and that it is an instance of how powerful the method of superposition is which has already been exemplified in the new proof of Euc. I, 8 which dispenses with the clumsy proposition Euc. I, 7.

2. Extension of a theorem of Salmon's.

In his book on Conics, Salmon gave the following theorem : "If A, B, C, D be any four points on a circle and O is any fifth point taken arbitrarily, and if we denote by BCD the area of the triangle BCD, etc., then

$$OA^2.BCD+OC^2.ABD=OB^2.ACD+OD^2.ABC."$$

Mookerjee has, in this paper, given first an extension of the above theorem by taking $2n$ points, instead of 4 points, on a circle. To arrive at the extended theorem, he considers initially the particular case when $n=3$, *i.e.*, the case of a hexagon $A_1A_2......A_6$ inscribed in a circle. This hexagon is divided into the following 6 quadrilaterals :

$$A_1A_2A_3A_4,\quad A_2A_3A_4A_5,\quad A_3A_4A_5A_6,$$
$$A_4A_5A_6A_1,\quad A_5A_6A_1A_2,\quad A_6A_1A_2A_3.$$

Now taking any point O in the plane and applying Salmon's theorem to these quadrilaterals, 6 equations of the type

$$OA_1^2.A_2A_3A_4+OA_3^2.A_1A_2A_4=OA_2^2.A_1A_3A_4+OA_4^2.A_1A_2A_3$$

are obtained. Adding these equations, an equation of the type

$$OA_1^2.\triangle_1+OA_3^2.\triangle_3+OA_5^2.\triangle_5=OA_2^2.\triangle_2+OA_4^2.\triangle_4+OA_6^2.\triangle_6$$

is obtained. It is then seen that each of the coefficients $\triangle_1,\ldots.,\triangle_6$ can be expressed as a difference of two triangles, *Eg.*, $\triangle_1=A_3A_4A_6-A_2A_4A_6$. The author then gives a rule by which these differences can be written down.

In the general case of a $2n$-gon, $(n>3)$, inscribed in a circle, the coefficients of OA_1^2, OA_2^2,......are seen to be differences of two $(2n-3)$-gons. A general rule is finally obtained by the author by means of which these differences can be written down.

The space analogue of Salmon's theorem is then obtained by Mookerjee, which is that if two tetrahedra O_1ABC and O_2ABC standing on the opposite sides of the same base ABC are inscribed in the same sphere and if O is any arbitrary point in space, then

$$\begin{aligned} OA^2.\ BCO_1O_2+OC^2.\ ABO_1O_2+OO_2^2.\ ABCO_1 \\ =OB^2.\ ACO_1O_2+OO_1^2.\ ABCO_2, \end{aligned}$$

where BCO_1O_2 etc. represent volumes of the tetrahedra BCO_1O_2 etc.

3. A note on elliptic functions.

In this paper Mookerjee gives a new proof of the addition-equation of the elliptic integrals of the first kind. Although various proofs of this equation existed, the one given here depends on the properties of the ellipse which, according to the author, it was natural to seek for. The proof briefly is this :

Let
$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

be a given ellipse and P be a point on any tangent

$$y=mx+\{a^2(1+m^2)-c^2\}^{\frac{1}{2}}$$

to the ellipse. Now two conics, one an ellipse and the other a hyperbola, can be drawn through P which are confocal with the given ellipse, namely the conics

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-c^2}=1 \text{ and } \frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-c^2}=1, \text{ where } \lambda>a>c>\mu.$$

Then the Cartesian coordinates of P satisfy

$$x^2=\lambda^2\mu^2/c^2,\quad y^2=-(\lambda^2-c^2)(\mu^2-c^2)/c^2.$$

From all these equations the following equation is deduced in a remarkable manner :

$$\frac{d\lambda}{\{(\lambda^2-c^2)(\lambda^2-a^2)\}^{\frac{1}{2}}}\pm\frac{d\mu}{\{(a^2-\mu^2)(c^2-\mu^2)\}^{\frac{1}{2}}}=0.$$

This is the differential equation satisfied by the elliptic coordinates of any point on any tangent of the given ellipse. Taking the $+$ sign and writing

$$\lambda=c\sin\phi,\quad \mu=c\sin\psi,\quad k=c/a,$$

the above equation is reduced to the following standard form (in the theory of elliptic integrals of the first kind) :

$$\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}}+\frac{d\psi}{(1-k^2\sin^2\psi)^{\frac{1}{2}}}=0. \tag{1}$$

And further, taking $m^2=k^2\sin^2\alpha-1$, the equation of tangent is transformed into

$$\cos\phi\cos\psi=\sin\phi\sin\psi\,(1-k^2\sin^2\alpha)^{\frac{1}{2}}+\cos\alpha. \tag{2}$$

Therefore (2) is the canonical form of the integral equation corresponding to the differential equation (1). (The addition-equation). Geometrical meaning of the symbols involved are assigned and discussed (ϕ is imaginary while ψ is real). The paper ends with an imaginary transformation suggested by this investigation.

4. **On the differential equation of a trajectory.**
7. **A general theorem on the differential equations of trajectories.**

The problem of determining the trajectory of a system of confocal ellipses was solved by the Italian mathematician Mainardi. Representing half the distance between the foci by h and the tangent of the angle of intersection by n, he gave the equation of the trajectory as

$$-2\tan^{-1}\sqrt{\left(\frac{h^2}{xM}-1\right)}+\log\frac{1-\sqrt{1-M/x}}{1+\sqrt{1-M/x}} = C, \tag{A}$$

where C is the constant of integration and M is given by the quadratic

$$(x^2+y^2+h^2)M = x(M^2+h^2).$$

Remarking that this form of the equation is so complicated that it would be a hopeless task to have to trace the curve from it, Mookerjee obtains in paper No. 4 a pair of remarkably simple equations which are equivalent to (A) and which, besides, admit an interesting geometrical interpretation. Writing

$$xM = h^2\cos^2\phi,\ C=2n\lambda,$$

where λ is a new constant, and substituting in (A) he gets after clever manipulations the pair of equations in question as

$$\left.\begin{aligned} x &= h\cos\phi.\cosh n(\lambda+\phi), \\ y &= h\sin\phi.\sinh n(\lambda+\phi). \end{aligned}\right\} \tag{B}$$

That is to say, the required trajectory is the locus of the point (x, y) defined by (B).

Regarding the geometrical interpretation of (B), let A, A', be the foci and O the middle point of AA', so that $OA=h$. On AA' as diameter describe a circle, draw a radius OB of the circle making angle $AOB=\phi$ and draw BC perpendicular to OA. Now construct a hyperbola CP having its centre at O and its transverse and conjugate axes equal to OC and BC respectively. Take the point P on the hyperbola so that the area of the hyperbolic sector OCP is $n(\lambda+\phi)$ times the area of the triangle OCB. Then the author shows that P is a point on the required trajectory, *i.e.*, a point having the coordinates (B). Thus the point P can be determined geometrically corresponding to any point B on the circle.

In paper No. 7 Mookerjee takes into consideration the method by which Mainardi obtained equation (A). This method, which is reproduced in Boole's Differential Equations, consists in first eliminating a and b between the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1,\quad a^2-b^2 = h^2,\quad \frac{dy}{dx} = \left(n\frac{y}{b^2}-\frac{x}{a^2}\right)\Big/\left(\frac{y}{b^2}+n\frac{x}{a^2}\right)$$

and then integrating the resulting equation. Remarking that this process is by no means less complicated than the equation (A), Mookerjee undertakes to enquire whether the process may be materially simplified. This leads him to formulate a general theorem on the differential equations of trajectories. This general theorem is however a bit too lengthy to be reproduced here. It may however be roughly stated as follows :

Whenever the coordinates of any point on a curve can be expressed by means of a single variable parameter, that is, when the curve is unicursal, the coordinates of the corresponding point on the trajectory may be similarly expressed.

Applications of this theorem, including Mainardi's problem, are then given.

5. On Monge's differential equation to all conics.

10. Remarks on Monge's differential equation to all conics.

11. The geometric interpretation of Monge's differential equation to all conics.

French mathematician G. Monge was the first to have obtained in 1810 the differential equation to all conics, *i.e.*, to curves represented by the general equation of the second degree in x, y. Denoting the differential coefficients of y with respect to x of the first, second, third, fourth and fifth orders by p, q, r, s and t, respectively, the differential equation in question is

$$9qt^2-45qrs+40r^3 = 0. \tag{C}$$

G. Boole in his book on Differential Equations (first edition, 1859) quoted the result (C) and made the following remark about it : "But here our powers of geometrical interpretation fail, and results such as this can scarcely be otherwise useful than as a registry of integrable forms."

Subsequently two attempts were made to interpret geometrically the equation (C). The first attempt was made in 1877 by Lieut. Col. Cunningham whose interpretation was that the "eccentricity of the osculating conic of a given curve is constant all around the latter." The second attempt was made in 1886 by Prof. Sylvester whose interpretation was that the equation (C) "is satisfied at the sextactic points of any given curve."

In paper No. 5 Mookerjee discusses in detail the questions of derivation of the differential equation (C) from the integral equation of the conic and the geometrical interpretation of (C) with reference to Boole's remark and Sylvester's interpretation, as mentioned above. As regards derivation, the author himself gives a method which can be applied with ease to any equation of the second degree in x, y. It consists of first expressing y as an explicit function of x and then proceeding to differentiation. As a matter of fact, he obtains by his method the following equation equivalent to (C)

$$\left(\frac{d}{dx}\right)^3\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0.$$

He also suggests methods of integrating equation (C). With regard to geometric interpretation, he carefully examines the one given by Sylvester and concludes that it cannot be the interpretation as contemplated by Boole.

The paper No. 10 was published in consequence of a letter written by Cunningham to the Asiatic Society of Bengal referring to paper No. 5 of Mookerjee and drawing attention to the fact that he had already given a geometric interpretation of (C). In paper No. 10 Mookerjee critically examines Cunningham's interpretation and rejects it on the ground that it does not satisfy the tests of a true geometrical interpretation of a differential equation. In fact he shows by actual calculation that Cunningham's interpretation was not the interpretation of the Mongian but of one of its first five independent first integrals. Later Cunningham acknowledged substantially the correctness of this criticism.

Finally in paper No. 11 Mookerjee gives his own interpretation of the Mongian (C) which depends on the following consideration : Consider the conic of closest contact at any point P of a given curve If O be the centre of the conic, O is called the centre of aberrancy, the length OP is called the radius of aberrancy and its reciprocal is called the index of aberrancy of the given curve at P. These notions are analogous to what are known as the centre of curvature, the radius of curvature and the curvature of a curve. Also the locus of O as P moves along the given curve is called the aberrancy curve of the given curve. It is seen that if (x, y) be the coordinates of the point P and (α, β) those of the point O, then

$$\alpha = x - \frac{3qr}{3qs-5r^2}, \quad \beta = y - \frac{3q(pr-3q^2)}{3qs-5r^2}, \tag{D}$$

where, as already said, p, q, r, s are the successive differential coefficients. (An account of the theory of aberrancy was given by Transon in Liouville's Journal)

With reference to the above ideas, Mookerjee finds out by ingenious process that the radius of curvature ρ of the aberrancy curve is given by

$$\rho = \frac{\{r^2+(pr-3q^2)^2\}^{\frac{3}{2}}}{q(3qs-5r^2)^3}(9qt^2-45qrs+40r^3).$$

Thus his geometrical interpretation of (C) is that "the radius of curvature of the aberrancy curve vanishes at every point of every conic."

9. On the differential equation of all parabolas.

In paper No. 5 Mookerjee gets also, by his own method, the differential equation of all parabolas in the form

$$\left(\frac{d}{dx}\right)^2\left[\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}\right] = 0$$

which when developed may be written in the then known form

$$3qs-5r^2 = 0, \tag{E}$$

where, as already stated in paper No. 5, p, q, r, s are the successive differential coefficients of y with respect to x.

In this paper No. 9 Mookerjee gives the geometrical interpretation of (E). To do this he uses the theory of aberrancy as stated in paper No. 11 and finds out by an ingenious process that the index of aberrancy I for a parabola is given by

$$I = \frac{3qs-5r^2}{3q\{r^2+(pr-3q^2)^2\}^{\frac{1}{2}}}.$$

Thus his geometrical interpretation of (E) is that "the index of aberrancy vanishes at every point of every parabola."

16. On a curve of aberrancy.

In this paper Mookerjee investigates the aberrancy curve of the plane cubic of Newton's fourth class

$$y = ax^3+3bx^2+3cx+d \tag{1}$$

in which the diametral conic degenerates into the line at infinity. Using formula (D) in the theory of aberrancy as stated in paper No. 11 he shows that the aberrancy curve of (1) is another plane cubic of the same class

$$y = Ax^3+3Bx^2+3Cx+D, \tag{2}$$

where A, B, C, D are such that if

$$H = ac-b^2, \quad G = a^2d-3abc+2b^2$$

are the invariants of (1) and

$$H'=AC-B^2, \quad G'=A^2D-3ABC-2B^2$$

are the corresponding invariants of (2), then

$$H' = -kH, \quad G' = k^2G$$

where k is a numerical constant ($=125/64$). Therefore H^2/G is an invariant for both the cubics (1) and (2). It is also shown that the two cubics have only one point of intersection which is a point of inflexion for both.

6. A memoir on plane analytical geometry.

This memoir is a big one at the beginning of which there is a table of contents showing the principal topics considered in the memoir. Regarding the object and scope of the paper Mookerjee says "It is my object in the present paper to bring together a number of theorems in the plane analytical geometry which have accumulated in my hands during my study of the subject.........I believe that either the theorems themselves or the methods of establishing them are original ; ...."

The different sections of the paper are to a great extent practically independent of each other and there are 32 of them. We shall merely quote here a theorem given by the author towards the end of the paper with his remark thereon :

"If from any point P tangents are drawn to the conic $S\equiv x^2/a^2+y^2/b^2-1=0$ and P is constrained to move on the curve $F(x, y)=0$, the locus of the middle point of the polar chord of P with respect to S is

$$F\left(\frac{x}{1+S}, \frac{y}{1+S}\right) = 0\text{''}$$

This result "is an immediate consequence of a new method which I propose to call Method of Elliptic Inversion."

8. On Poisson's integral

In this paper Mookerjee re-establishes Poisson's remarkable relation of the definite integrals

$$\int_0^\pi \frac{\sin^p x \, dx}{(1-2a\cos x+a^2)^n} = \frac{1}{(1-a^2)^{2n-p-1}} \cdot \int_0^\pi \frac{\sin^p x \, dx}{(1-2a\cos x+a^2)^{1+p-n}} \qquad \text{(F)}$$

by an interesting method and gives a geometrical interpretation to it. (The result holds when $a<1$ but if $a>1$, $1-a^2$ is to be replaced by its negative).

Denoting the left-hand side of (F) by y, Poisson first obtained the differential equation

$$(1-a^2)\frac{d^2y}{dx^2}+\left\{\frac{p+1}{a}-a(4n+1-p)\right\}\frac{dy}{dx}-2n(2n-p)y=0$$

and then he used this equation to demonstrate (F). His proof is however very abstruse. Mookerjee on the other hand bases his proof on the indefinite integral obtained from the left-hand side of (F) by omitting the limits. This method has the advantage, he points out, of showing how the indefinite integral may be evaluated.

Regarding the geometrical interpretation of (F), let APA' be a semi-ellipse on which P is any point and of which AA' is the major axis, C the centre and S a focus ; also let AQA' be a semi-circle on the same side of the axis as P. Join PS, draw PM at right angles to AA' meeting the semi-circle in Q and join QC. Further, let the angles ASP and ACQ be denoted by θ and u respectively. The author then takes the relation (as in the case of elliptic motion).

$$\cos\theta = \frac{\cos u - e}{1 - e\cos u}$$

from which he derives in a remarkable manner the transformation

$$\int \frac{\sin^p\theta\, d\theta}{(1+e\cos\theta)^n} = \frac{1}{(\sqrt{1-e^2})^{2n-p-1}} \int \frac{\sin^p u\, du}{(1-e\cos u)^{p-n+1}}\,.$$

Finally taking $e = -2a/(1+a^2)$, he obtains a relation equivalent to (F).

12, 13. Some applications of elliptic functions to problems of mean values. First and Second papers.

In the first of these two papers, Mookerjee establishes the following theorems :

1. The average area common to an ellipse and a concentric circle of variable radius which always intersects it is

$$\frac{1}{3}\,\frac{b}{a-b}\left\{\pi(3a^2-b^2) - al\right\}$$

where a, b are the semi axes and l is the perimeter of the ellipse. Here

$$l = \int_0^{\pi/2} (1 - e^2 \sin^2\xi)\, d\xi$$

is the complete elliptic integral of the second kind with eccentricity e for modulus.

II. The average value of the angle of intersection of an ellipse and a concentric circle of variable radius which always intersects it is

$$\frac{1+\lambda^2}{\lambda(1-\lambda)}\,P - \frac{2\lambda}{1-\lambda}\,F,$$

where λ is the ratio of the axes ($=b/a$) and F, P denote complete elliptic integrals of the first and third kind respectively, the modulus being $(1-\lambda^2)$ and the parameter $(1-\lambda^2)/\lambda^2$.

There are two other theorems of similar nature. In all these theorems the method of proof is that if A is the average value and P a particular value, then

$$(a-b)A = \int_b^a P\, dr,$$

where a, b are the semi-axes of the ellipse and r the variable radius of the circle.

The second paper with the same title is devoted to a discussion of the corresponding space analogue by taking ellipsoid and sphere in place of ellipse and circle.

14. On Clebsch's transformation of the hydrokinetic equations.

15. Note on Stoke's theorem and hydrokinetic circulation.

In paper No. 14 Mookerjee considers the hydrokinetic equations in the three cases : (1) Irrotational motion, (2) steady rotational motion and (3) general rotational motion. He shows how the method of Clebsch's transformation to the third case may be materially simplified.

In paper No. 15 Mookerjee gives a new proof of Stoke's formula for hydrokinetic circulation by the application of Clebsch's transformation.

17. Mathematical Notes in London Educational Times.

In these notes, mathematical questions were proposed and their solutions were given. A large number of mathematicians, including Profs. Clifford, Cayley, Sylvester, Knowles, Neuberg, Schoute, and also some Indians used to contribute in these notes. Some of the questions remained unsolved for many years. Asutosh contributed about a dozen questions and solutions. In some of them he solved questions proposed by others and in the remaining he proposed questions which were or were to be solved by others. Illustrations are given below.

The following is a question proposed by Asutosh and solved by others :

Two imperfectly elastic particles, whose coefficient of elasticity is λ are projected with equal velocities in opposite directions, at right angles to the line of force at the point of projection ; assuming the force to vary inversely as the square of the distance, and the second impact to take place midway between the point of projection and the centre of forces, show that

$$e_1 = \pm(2-3\lambda^2)/\lambda^2, \; e_2 = \pm(3-4\lambda^2),$$

where e_1 is the eccentricity of the first orbit and e_2 that of the orbit between the first and the second impacts.

The following is a question proposed by some one and solved by Asutosh :

Prove that the four points of intersection of the opposite sides of an octagon inscribed in a conic are collinear.

CONTRIBUTION TO THE THEORY OF GENERAL GEOMETRODYNAMICS II(*)

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1. Prerequisites

Let us consider the Einstein-Maxwell equations

$$R^{\nu}_{\lambda} - \tfrac{1}{2} R \delta^{\nu}_{\lambda} = P^{\nu}_{\lambda} \quad {}^{*)} \tag{1.1)a}$$

where the momentum energy tensor P^{ν}_{λ} is defined by

$$P^{\nu}_{\lambda} = \mu u_{\lambda} u^{\nu} + F^{\nu}_{\lambda} + P \delta^{\nu}_{\lambda} . \tag{1.1)b}$$

Here μ is the rest mass, u_{λ} is the relativistic velocity unit vector, p is the pressure (which must not be confused with a hydrodynamical pressure). F^{ν}_{λ} is the momentum energy tensor of the electromagnetic field. Its Weierstrass characteristic is [(11) (11)]. If we denote by $+\frac{M^2}{2}$ and $-\frac{M^2}{2}$ its eigenvalue then

$$F^{\nu}_{\lambda} = \frac{M^2}{2} \left[-\overset{1}{u}_{\lambda} \underset{1}{u}^{\nu} - \overset{2}{u}_{\lambda} \underset{2}{u}^{\nu} + \overset{3}{u}_{\lambda} \underset{3}{u}^{\nu} + \overset{4}{u}_{\lambda} \underset{4}{u}^{\nu} \right] . \tag{1.1)c}$$

Here $\underset{1}{u}^{\nu}, \ldots, \underset{4}{u}^{\nu}$ are eigenvectors and $\overset{1}{u}_{\lambda}, \ldots \overset{4}{u}_{\lambda}$ constitute the dual system

$$\underset{a}{u}^{\nu} \overset{a}{u}_{\lambda} = \delta^{\nu}_{\lambda} \tag{1.2}$$

If one considers in (1.1) the tensor P^{ν}_{λ} as given, then its geometric properties enable us to express the ten unknown components of the metric tensor $g_{\lambda\mu}$ in terms of N scalars **)

$$10 > N \geqslant 3$$

The system (1.1) represents then ten differential equations for N unknowns. This system can be solved by Cauchy problem provided that certain conditions are satisfied.

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*) Lower case Greek (Latin) indices have the range I, . . . , IV (1, . . ., 4) unless stated explicitly otherwise.

**) *Cf.* [6] [7], [8], [9]

If on the other hand we consider the metric tensor $g_{\lambda\mu}$ as known, then we know also its contracted curvature tensor R^{ν}_{λ} and $R= R^{\lambda}_{\lambda}$ so that the left hand side of (1.1)*a* is known. In this case the elements of P^{ν}_{λ} *i.e.*,

$$\mu,\ u_{\lambda},\ \underset{a}{u}^{\nu},\ \overset{a}{u}_{\lambda},\ p,\ M^2 \tag{1.3}$$

may be expressed as algebraic concomitants of R^{ν}_{λ} (*i.e.*, as differential concomitants of $g_{\lambda\mu}$ of second order). *), **)

If we limit ourselves to real eigenvalues and to the first two classes of the electromagnetic field then R^{ν}_{λ} $\left(\text{as well as } P^{\nu}_{\lambda}\right)$ can acquire only the following Weierstrass characteristics

(1.4) [1111], [112], [(11) 11], [(111) 1].

In all these cases the elements (1.3), the electromagnetic field as well as the electric charge may be explicitly expressed as concomitants of R^{ν}_{λ} ***) *In the sequel we shall deal only with the characteristic* [1111]. Denoting by $\rho_1, \rho, \rho', \rho''$ the eigenvalues of R^{ν}_{λ} and putting

(1.5)

$$(a)\quad \tau=\rho-\rho_1, \qquad \tau'=\rho'-\rho_1,\ \tau''=\rho''-\rho_1$$

$$(b)\quad x^2=\frac{\tau'\tau''}{\tau(\tau-\tau'-\tau'')} > 0$$

we get $(a)\quad \tau=M^2$

(1.6)

$$(b)\quad \tau' = \frac{M^2-\mu}{2}+\sqrt{\left(\frac{M^2-\mu}{2}\right)^2-M^2\mu x^2}$$

$$(c)\quad \tau'' = \frac{M^2-\mu}{2}-\sqrt{\left(\frac{M^2-\mu}{2}\right)^2-M^2\mu x^2}$$

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*) *Cf.* [1], [2] for the case of the second and first class of the electromagnetic field, [3] for the case of the third class. These three papers deal with real eigenvalues of R^{ν}_{λ} only. The case of complex eigenvalues is considered in [4], [6].

**) As a matter of fact the electromagnetic field itself as well as the electric charge are concomitants of $g_{\lambda\mu}$. *Cf.* for different classes [1], [2], [3], [5].

***) *Cf.* [1, [2], [5]

so that

(1.6) $(d)\ \tau'+\tau''=M^2-\mu \qquad (e)\ \tau'\tau''=M^2\mu x^2$

and

(1.6) $(\acute{e})\ \mu=\tau-(\tau'+\tau'')$.

Moreover if we denote by Ω' and Ω'' the expressions

(1.7)
$$(a)\ \Omega' \overset{\text{def}}{=} \frac{\tau'}{\tau-\tau'-\tau''}(\tau-\tau')(\tau''-\tau')$$
$$(b)\ \Omega'' \overset{\text{def}}{=} \frac{\tau''}{\tau-\tau'-\tau''}(\tau-\tau'')(\tau'-\tau'')$$

it turns out that the eigenvector $\begin{cases} v'^{\nu} \\ v''^{\nu} \end{cases}$

belonging to the eigenvalue $\begin{cases} \rho' \\ \rho'' \end{cases}$ is timelike if and only if

(1.8)
$$\begin{matrix} (a) \\ (b) \end{matrix}\begin{cases} \Omega' < 0 & \Omega'' > 0 \\ \Omega'' < 0 & \Omega' > 0 \end{cases}$$

The pressure p is given by

(1.6) $(f)\ 2p=-(\rho'+\rho'')$

(so that ρ_1 does not appear in this expression). It is possible also to express $u, \underset{3}{u}, \underset{4}{u}$ in terms of the eigenvalues and eigenvectors. However we shall not need in the sequel these expressions and therefore we do not introduce them.

2. Problem to be solved

Let us assume that R^{ν}_{λ} is known and its characteristic is [1111]. Then its eigenvalues $\rho_1, \rho, \rho', \rho''$ and the corresponding eigenvectors $\underset{1}{v}, v, v', v''$ are also known. Therefore we may apply the results of the previous section and obtain all elements (1.3) as concomitants of R^{ν}_{λ}. It is a priori evident that one of the most important problems of this method is to decide which eigenvalue should be designed by ρ_1, ρ, ρ' or ρ''. In [5] where we were concerned with such problems we displayed a method which indicates how to decide which one of the eigenvalues is ρ_1, ρ, ρ' or ρ''. The decision was relatively a simple one, because we considered only the tensor (1.1).

In this paper we shall generalize this method for the case of the momentum energy tensor

(2.1)
$$P^{\nu}_{\lambda}(\epsilon_1, \epsilon_2, \epsilon_3) \overset{\text{def}}{=} \epsilon_1\mu u_{\lambda}u^{\nu}+\epsilon_2 F^{\nu}_{\lambda}+\epsilon_3 p\,\delta^{\nu}_{\lambda}$$
$$\epsilon_1\epsilon_2\epsilon_3=\pm 1$$

admitting even negative mass, *i.e.*, $\epsilon_1=-1$ and negative pressure $\epsilon_3=-1$. In contradistinction to the situation in [5] where we admitted only one possibility for the momentum energy tensor, namely (1.1)*b* we have here eight possibilities according to the values ϵ_1, ϵ_2, ϵ_3. It turns out that one has to refine the method used in [5]. We shall display here in detail this refinement.

Problem to be solved. Assume that the metric tensor $g_{\lambda\mu}$ is given. Construct its corresponding contracted curvature tensor R^{ν}_{λ} and its four different real eigenvalues *) $\rho_1, \rho, \rho', \rho''$. Denote by ρ_1 the eigenvalue which is not given by (1.6), by ρ_4 the eigenvalue which leads to the time-like eigenvector and by ρ_2, ρ_3 the remaining eigenvalues. The problem consists of

(1) Finding one-to-one correspondence between ρ_a and $\rho_1, \rho, \rho', \rho''$ which appear in (1.6) **)

(2) Finding in terms of ρ_a necessary and sufficient condition that the field equations

$$R^{\nu}_{\lambda} - \tfrac{1}{2}R\delta^{\nu}_{\lambda} = P^{\nu}_{\lambda}(\epsilon_1, \epsilon_2, \epsilon_3) \tag{2.2}$$

be satisfied, where R^{ν}_{λ} is the contracted curvature tensor of $g_{\lambda\mu}$.

(3) Expressing the elements (1.3) as algebraic concomitants of R^{ν}_{λ} (*i.e.*, differential concomitants of order two of $g_{\lambda\mu}$)

(4) Investigating the influence of ρ_4 on the partial problems (1), (2), (3).

This problem has been solved for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ in [5]. The results (which depend on the choice of the ϵ'*s*) are introduced in the first section, equations (1.6), (1.8) where either

$$\rho_2 > \rho_1 > \rho_3 > \rho_4 \tag{2.4a}$$

or

$$\rho_2 > \rho_4 > \rho_3 > \rho \tag{2.4b}$$

and

$$\tau = \rho_2 - \rho_1 \quad \tau' = \rho_3 - \rho_1 \quad \tau'' = \rho_4 - \rho_1. \tag{2.4c}$$

The remaining elements (2.1) are independent of the choice of ϵ_1, ϵ_2, ϵ_3 and therefore are the same as for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. Because we shall not need them in the sequel we do not introduce them here, keeping in mind that their analytic expressions in terms of ρ_1, ρ_2. ρ_3, ρ_4, as displayed in [5] is always the same.

Once we identified the ρ'*s* in the general case of ϵ_1, ϵ_2, ϵ_3 we can use always the formulae (1.6) and (1.8) with the proviso that μ, M^2, p are replaced by $\epsilon_1\mu$, $\epsilon_2 M^2$, $\epsilon_3 p$.

From the above statement about the elements (1.3) it follows easily, that after having satisfied the requirements (1) and (2) we may omit the elaboration of 3) because for the elements $\epsilon_1\mu$, $\epsilon_2 M^2$, $\epsilon_3 p$ it is described by (1.6) and (1.8) while for the remaining elements it is the same for any choice of ϵ_1, ϵ_2, ϵ_3. Therefore we shall skip in the sequel the requirement 3).

---

*) We are limiting ourselves to a real characteristic [1111].

**) As far as the identification of $\rho_1, \rho, \rho', \rho''$ is concerned we know only that they satisfy (1.6) where μ, M, p are not known.

From now on we shall assume

(1) *that* μ, M^2, p *are always positive and*

(2) *that in* (1.6) *we replaced*

$$\mu,\ M^2,\ p \text{ by } \epsilon_1\mu,\ \epsilon_2 M^2,\ \epsilon_3 p.$$

Then it follows from (1.5)*b* and (1.6) that for all ϵ'*s*

$$(2.5)\qquad \left(\frac{\epsilon_2 M^2-\epsilon_1\mu}{2}\right)^2-\epsilon_1\epsilon_2 M^2\mu x^2=\left(\frac{\tau'-\tau''}{2}\right)^2>0\ ^{*)}$$

while (1.7) and (1.8) yield

$$(2.6)a\qquad \Omega'\Omega''=-\tau'\tau''(\tau-\tau')(\tau-\tau'')(\tau'-\tau'')^2(\tau-\tau'-\tau'')^{-2}<0 \text{ so that}$$

$$(2.6)b\qquad \tau'\tau''(\tau-\tau')(\tau-\tau'')>0$$

We shall very often use these inequalities.

3. Positive mass. Necessary and sufficient conditions

Our problem is purely an algebraic one. This will be stressed by the following.

Agreement (3.1). *Whenever we consider equations* (2.2) *we tacitly assume that there is a metric tensor* $g_{\lambda\mu}$ *satisfying these equations with* R^{ν}_{λ} *as its contracted curvature tensor.*

The importance of this Agreement will be seen later.

Theorem (3.1). *In order that*

(1) *the contracted curvature tensor* R^{ν}_{λ} *of* $g_{\lambda\mu}$ *be of characteristic* [1111],

(2) *the tensor* $R^{\nu}_{\lambda}-\frac{1}{2}R\delta^{\nu}_{\lambda}$ *satisfy* (2.2) *where* $P^{\nu}_{\lambda}(\epsilon_1, \epsilon_2, \epsilon_3)$ *is given by* (2.1) *for*

$$(3.1)\qquad (a)\ \ \epsilon_1=\epsilon_2=1 \qquad (b)\ \ \mu<M^2$$

it is necessary and sufficient that the following conditions are satisfied

$$(3.2)\qquad (a)\ \ (\rho'=\rho_4)^{**}\quad \rho_2>\rho_4>\rho_3>\rho_1$$

$$(b)\ \ \rho_2+\rho_1>\rho_3+\rho_4\begin{cases}<0\\>0\end{cases}\text{for } \epsilon_3=\begin{cases}1\\-1\end{cases}$$

$$(c)\ \ \rho_3+\rho_4>2\rho_1.$$

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*) Comparing (2.5) with (1.6) *b*) *c*) we see that $\surd$ in the latter equations must be taken positively. This remark is important for later purposes.

**) This eqation is in parentheses because it is not needed for the computation of the elements (1.3) of P^{ν}_{λ}. One needs it only if by some reasons one wants to express (1.6) *b*), *c*), in terms of ρ_3, ρ_4, ρ_1.

Proof. Assume that conditions (1), (2) are satisfied. Then we obtain from (2.5).

$$\left(\left(\frac{M^2-\mu}{2}\right)^2-M^2\mu\chi^2\right)^{\frac{1}{2}}=\frac{M^2-\mu}{2}-\sigma^{*)},\quad 0<2\sigma<M^2-\mu.$$

Hence

(3.3) $(a)\ \tau'=M^2-\mu-\sigma,\ (b)\ \tau''=\sigma,\ (c)\ \tau'-\tau''=M^2-\mu-2\sigma,$
$(d)\ \tau-\tau'=\mu+\sigma$

and consequently

(3.4) $(a)\ \tau'>0,\ (b)\ \tau''>0\ (c)\ \tau'-\tau''>0,\ (d)\ \tau-\tau'>0\ (e)\ \tau-\tau''>0$**).

The inequalities (3.4)*cde* require

(3.5)*a* $\tau>\tau'>\tau''.$

Because of $\epsilon_2=1$ and (1.6)*a* we have $\tau>0$***).

and this inequality together with (3.4)*ab* leads to

(3.5)*b* $\rho>\rho'>\rho''>\rho_1$

From the method displayed in [5] we know that either ρ' or ρ'' but not ρ is the eigenvalue of the time-like eigenvector so that we must have

(3.6)*a* $\rho=\rho_2.$

An easy inspection shows that (1.8)*a* is satisfied for (3.6)*a* and

(3.6)*b* $\rho'=\rho_4$

while (1.8)*b* is not satisfied by (3 6)*a* and $\rho''=\rho_4$. The equations (3.6) and the inequalities (3.5)*b* lead to (3.2)*a*. The first inequality (3.2)*b* is a consequence of $\epsilon_1=1$, and (1.6)*é*. The second inequality (3.2)*b* follows from (1.6)*f* while the last inequality is equivalent to (3.1)*b* (*Cfr*(1.6)*aé*).

If (3.2) hold the method displayed in [5] shows that not only μ, M^2 and p (given by (1.6)*aef*) but also the remaining elements (1.3) may be found (if we know R^{ν}_{λ}) so that (2.2), (2.1) and (3.1) are satisfied. According to Agreement (3.1) R^{ν}_{λ} is the contracted curvature tensor of a metric tensor.

Theorem (3.2). *In order that*

(1) *the contracted curvature tensor* R^{ν}_{λ} *of* $g_{\lambda\mu}$ *be of characteristic* [1111],

(2) *the tensor* $R^{\nu}_{\lambda}-\frac{1}{2}R\delta^{\nu}_{\lambda}$ *satisfy* (2.2) *where* $P^{\nu}_{\lambda}(\epsilon_1, \epsilon_2, \epsilon_3)$ *is given by* (2.1) *for*

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*) It is irrelevant for our purposes whether we use this equation or

$$\left(\left(\frac{M^2-\mu}{2}\right)^2-M^2\mu\chi\right)^{\frac{1}{2}}=\frac{\mu-M^2}{2}+\lambda,\qquad \lambda>0$$

Similar remark holds for combination of ϵ_1 and ϵ_2

**) (3.4)*e* follows from (2 6)*b* and (3.4)*abd*;

***) This follows also from (3.4)*ad*

$$(a)\quad \epsilon_1=\epsilon_2=1 \qquad (b)\quad \mu>M^2 \tag{3.7}$$

it is necessary and sufficient that the following conditions are satisfied

$$\begin{aligned} &(a)\quad \rho''=\rho_4 \quad \rho_2>\rho_1>\rho_3>\rho_4 \\ &(b)\quad \rho_2+\rho_1>\rho_3+\rho_4 \begin{cases}<0\\>0\end{cases} \text{for } \epsilon_3=\begin{cases}1\\-1\end{cases} \\ &(c)\quad \rho_3+\rho_4<2\rho_1. \end{aligned} \tag{3.8}$$

Proof. Assume that conditions (1), (2) are satisfied. Then we obtain from (2.5)

$$\left(\left(\frac{M^2-\mu}{2}\right)^2-\mu M^2\chi^2\right)^{\frac{1}{2}}=\left|\frac{M^2-\mu}{2}\right|-\sigma,\ 0<2\sigma<|M^2-\mu|.$$

Hence

$$\begin{aligned} &(a)\quad \tau'=-\left|\frac{M^2-\mu}{2}\right|+\left|\frac{M^2-\mu}{2}\right|-\sigma=-\sigma \\ &(b)\quad \tau''=-|M^2-\mu|+\sigma \quad (c)\quad \tau'-\tau''=+|M^2-\mu|-2\sigma \\ &(d)\quad \tau-\tau'=M^2+\sigma \end{aligned} \tag{3.9}$$

and consequently

$$(a)\quad \tau'<0,\quad (b)\quad \tau''<0,\quad (c)\quad \tau'-\tau''>0,\quad (d)\quad \tau>\tau',\quad (e)\quad \tau>\tau'' \text{ *)} \tag{3.10}$$

The inequalities (3.10) (*c*) (*d*) (*e*) require (3.5)*a*. Because of $\epsilon_2=1$ and (1.6)*a* we have $\tau>0$ and this inequality in conjunction with (3.10)*ab* and (3.5) yields

$$\rho>\rho_1>\rho'>\rho''. \tag{3.11)a}$$

An easy inspection shows that (1.8)*b* is satisfied by

$$\rho''=\rho_4 \tag{3.11)b}$$

while (1.8)*a* is not satisfied by $\rho'=\rho_4$. The equations (3.6)*a*, (3.11)*b* and the inequalities (3.11)*a* lead to (3.8)*a*. The remaining part of the proof follows the pattern of the pre-previous proof.

Theorem (3.3). *In order that*

(1) *the contracted curvature tensor* R^{ν}_{λ} *of* $g_{\lambda\mu}$ *be of characteristic* [1111],

(2) *the tensor* $R^{\nu}_{\lambda}-\frac{1}{2}R\delta^{\nu}_{\lambda}$ *satisfy* (2.2) *where* P^{ν}_{λ} $(\epsilon_1, \epsilon_2, \epsilon_3)$ *is given by* (2.1) *for*

$$\epsilon_1=-\epsilon_2=1 \tag{3.12}$$

it is necessary and sufficient that the following conditions are satisfied

$$\begin{aligned} &(a)\quad \rho''=\rho_4 \quad \rho_3>\rho_1>\rho_2>\rho_4 \\ &(b)\quad \rho_2+\rho_1>\rho_3+\rho_4 \begin{cases}<0\\>0\end{cases} \text{for } \epsilon_3=\begin{cases}1\\-1\end{cases}. \end{aligned} \tag{3.13}$$

Proof. Assume that conditions (1), (2) are satisfied. Then we have from (2.5)

$$\left(\left(\frac{-M^2-\mu}{2}\right)^2+\mu M^2\chi^2\right)^{\frac{1}{2}}=\frac{M^2+\mu}{2}+\sigma \quad \sigma>0.$$

---

*) (3.10)*e* follows from (2.6)*b* and (3.10)*abd*

Hence

(3.14) (*a*) $\tau'=-\frac{M^2+\mu}{2}+\frac{M^2+\mu}{2}+\sigma=\sigma$ (*b*) $\tau''=-(M^2+\mu)-\sigma$

(*c*) $\tau''-\tau'=-(M^2+\mu)-2\sigma$ (*d*) $\tau''-\tau=-(M^2+\mu)-\sigma+M^2=-(\mu+\sigma)$

and consequently

(3.15) (*a*) $\tau'>0$, (*b*) $\tau''<0$, (*c*) $\tau'>\tau''$, (*d*) $\tau>\tau''$, (*e*) $\tau<\tau'$.*)

The inequalities (3.15) (*c*) (*d*) (*e*) require

(3.16) $$\tau'>\tau>\tau''.$$

Because of $\epsilon_2=-1$ and (1.6)*a* we have $\tau<0$ and this inequality in conjunction with (3.15)*ab* yields

(3.17) $$\rho' > \rho_1 > \rho > \rho''.$$

An easy inspection shows that (1.8)*b* is satisfied for $\rho=\rho_2$ and

(3.18) $$\rho''=\rho_4$$

while (1.8)*a* is not satisfied by $\rho'=\rho_4$. In this way we obtain via (3.6)*a* (3.18) and (3.17) the conditions (3.13)*a*. The remaining part of the proof follows the pattern of previous proofs.

4. Negative mass. Necessary and sufficient conditions

Theorem (4.1). *In order that*

(1) *the contracted curvature tensor* R^{ν}_{λ} *of* $g_{\lambda\mu}$ *be of characteristic* [1111],

(2) *the tensor* $R^{\nu}_{\lambda}-\frac{1}{2}R\delta^{\nu}_{\lambda}$ *satisfy* (2·2), *where* $P^{\nu}_{\lambda}(\epsilon_1,\epsilon_2,\epsilon_3)$ *is given by* (2.1) *for*

(4.1) $$-\epsilon_1=\epsilon_2=1$$

it is necessary and sufficient that the following conditions are satisfied

(4.2)
$$(a)\quad \rho'=\rho_4\quad \rho_4>\rho_2>\rho_1>\rho_3$$
$$(b)\quad \rho_2+\rho_1<\rho_3+\rho_4\begin{cases}<0\\>0\end{cases}\text{for } \epsilon_3=\begin{cases}1\\-1\end{cases}.$$

Proof. Assume that conditions (1), (2) are satisfied. Then we have from (2.5)

$$\left(\left(\frac{M^2+\mu}{2}\right)^2+M^2\mu\chi^2\right)^{\frac{1}{2}}=\frac{M^2+\mu}{2}+\sigma,\quad \sigma>0.$$

Hence

(4.3)
$$(a)\quad \tau'=M^2+\mu+\sigma,\quad (b)\quad \tau''=-\sigma,\quad (c)\quad \tau'-\tau''=M^2+\mu+2\sigma$$
$$(d)\quad \tau-\tau''=M^2+\sigma$$

---

*) (3.15)*e* follows from (3.15)*abd* and (2.6)*b*

and consequently

(4.4) $(a)\ \tau'>0,\ (b)\ \tau''<0,\ (c)\ \tau'>\tau'',\ (d)\ \tau>\tau'',\ (e)\ \tau'>\tau.$ *)

The inequalities (4.4) require

(4.5)*a* $\tau'>\tau>\tau''.$

Because of $\epsilon_2=1$ and (1.6)*a* we have $\tau>0$ **)

and this inequality in conjunction with (4.4)*ab* leads to

(4.5)*b* $\rho'>\rho_2>\rho_1>\rho''.$

An easy inspection shows that (1.8)*a* is satisfied by $\rho'=\rho_4$ while (1.8)*b* is not satisfied by $\rho''=\rho_4$. The equations

(4.6) $\rho'=\rho_4$

and (3.6)*a* lead in conjunction with (4.5)*b* to (4.2). The first inequality (4.2)*b* is a consequence of $\epsilon_1=-1$ and (1.6)*e*. The remaining part of the proof follows the pattern of previous Proofs.

Theorem (4.2). *In order that*

(1) *the contracted curvature tensor* R^{ν}_{λ} of $g_{\lambda\mu}$ *be of characteristic* [1111]

(2) *the tensor* $R^{\nu}_{\lambda}-\frac{1}{2}R\delta^{\nu}_{\lambda}$ *satisfy* (2.2) *where* P^{ν}_{λ} $(\epsilon_1, \epsilon_2, \epsilon_3)$ *is given by* (2.1) *for*

(4.7) $\epsilon_1=\epsilon_2=-1\quad \mu>M^2$

it is necessary and sufficient that the following conditions are satisfied

(4.8)

$(a)\ \rho'=\rho_4\quad \rho_4>\rho_3>\rho_1>\rho_2$

$(b)\ \rho_2+\rho_1<\rho_3+\rho_4 \begin{cases} <0 \\ >0 \end{cases} \text{for } \epsilon_3=\begin{cases} 1 \\ -1 \end{cases}$

$(c)\ \rho_3+\rho_4>2\rho_1.$

Proof. Assume that conditions (1), (2) are satisfied. Then we have from (2.5)

$$\left(\left(\frac{M^2-\mu}{2}\right)^2-M^2\mu\chi^2\right)^{\frac{1}{2}}=\frac{\mu-M^2}{2}-\sigma,\quad 0<\sigma<\frac{\mu-M^2}{2}.$$

Hence

(4.9) $(a)\ \tau'=\frac{\mu-M^2}{2}+\frac{\mu-M^2}{2}-\sigma=\mu-M^2-\sigma,\quad (b)\ \tau''=\sigma$

$(c)\ \tau'-\tau''=\mu-M^2-2\sigma,\qquad (d)\ \tau''-\tau=\sigma+M^2$

and consequently

---

*) (4 4)*e* follows from (4 4) *a*) *b*) *d*) and (2.6)*b*.

**) This follows also from (4 4) a) d).

(4.10) $(a)\ \tau'>0,\ (b)\ \tau''>0,\ (c)\ \tau'>\tau'',\ (d)\ \tau''>\tau,\ (e)\ \tau'>\tau$.\*)

The inequalities (4.10) require

(4:11) $(a)\ \tau'>\tau''>\tau.$

Because of $\epsilon_2=-1$ and (1.6)*a* we have $\tau<0$ and this inequality in conjunction with (4.10) (*a*) (*b*) leads to

(4.11) $(b)\ \rho'>\rho''>\rho_1>\rho_2.$

An easy inspection shows that (1.8)*a* is satisfied by

(4.12) $\rho'=\rho_4$

while (1.8)*b* is not satisfied by $\rho''=\rho_4$. The equations (4.12) and (3.6)*a* in conjunction with (4.11)*b* lead to (4.8)*a*. The remaining part of the proof follows the pattern of previous proofs.

Theorem (4.3). *In order that*

(1) *the contracted curvature tensor* $\overset{\nu}{\underset{\lambda}{R}}$ *of* $g_{\lambda\mu}$ *be of characteristic* [1111]

(2) *the tensor* $\overset{\nu}{\underset{\lambda}{R}}-\frac{1}{2}R\overset{\nu}{\underset{\lambda}{\delta}}$ *satisfy* (2.2) *where* $\overset{\nu}{\underset{\lambda}{P}}(\epsilon_1,\epsilon_2,\epsilon_3)$ *is given by* (2.1) *for*

(4.13) $\epsilon_1=\epsilon_2=-1\quad \mu<M^2$

it is necessary and sufficient that the following conditions are satisfied

(4.14)
$$(a)\ \rho''=\rho_4\quad \rho_1>\rho_3>\rho_4>\rho_2$$
$$(b)\ \rho_2+\rho_1\ <\ \rho_3+\rho_4\begin{cases}<0\\>0\end{cases} for\ \epsilon_3=\begin{cases}1\\-1\end{cases}$$
$$(c)\ \rho_3+\rho_4<2\rho_1.$$

Proof. Assume that the conditions (1), (2) are satisfied. Then we have from (2.5)

$$\left(\left(\frac{\mu-M^2}{2}\right)^2-M^2\mu\chi^2\right)^{\frac{1}{2}}=\frac{M^2-\mu}{2}-\sigma,\ 0<2\sigma<M^2-\mu.$$

Hence

(4.15)
$$(a)\ \tau'=\frac{\mu-M^2}{2}+\frac{M^2-\mu}{2}-\sigma=-\sigma$$
$$(b)\ \tau''=-[M^2-\mu-\sigma].\quad (c)\ \tau'-\tau''=M^2-\mu-2\sigma$$
$$(d)\ \tau''-\tau=\mu+\sigma$$

and consequently

(4.16) $(a)\ \tau'<0,\ (b)\ \tau''<0,\ (c)\ \tau'>\tau'',\ (d)\ \tau''>\tau,\ (e)\ \tau'>\tau.$ **)

The inequalities (4.16) require

(4.17)*a* $\tau'>\tau''>\tau.$

Because of $\epsilon_2=-1$ and (1.6)*a* we must have $\tau<0$ and this inequality in conjunction with (4.17)*a* leads to

(4.17)*b* $\rho_1>\rho'>\rho''>\rho.$

An easy inspection shows that (1.8)*b* is satisfied by

(4.18) $\rho''=\rho_4$

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*) (4.10)*e* follows from (4.10) (*a*) (*b*) (*d*) and (2.6)*b*.

**) (4.16)*e* follows from (4.16) (*a*) (*b*) (*d*) and (2.6*b*).

while (1.8)*a* is not satisfied by $\rho'=\rho_4$. The equations (4.18) and (3.6)*a* lead in conjunction with (4.17)*b* to (4.14)*a*. The remaining part of the proof follows the pattern of previous proofs.

5. Eigenvalue ρ_4 and its significance

There is only one eigenvalue, namely ρ_4, which leads to the time-like eigenvector of $\overset{\nu}{\underset{\lambda}{R}}$. Hence if $\overset{\nu}{\underset{\lambda}{R}}$ is given (as it is in our case) we know without ambiguity which one of its four eigenvalues is called ρ_4. In contradistinction to ρ_4 the correspondence between the remaining three eigenvalues and the symbols ρ_1, ρ_2, ρ_3 is not defined by $\overset{\nu}{\underset{\lambda}{R}}$ alone. In our case we imposed some conditions on the eigenvalues and these conditions change the situation. In the sequel we shall investigate the influence to the just mentioned correspondence between the three remaining eigenvalues and the symbols ρ_1, ρ_2, ρ_3. To this effect we introduce first some definitions which will be needed later.

Definition (5.1). *The conditions*

(5.1)
(*a*) $\epsilon_1=\epsilon_2=1, \quad \mu<M^2$ (Theorem (3.1))
(*b*) $\epsilon_1=\epsilon_2=1, \quad \mu>M^2$ (Theorem (3.2))
(*c*) $\epsilon_1=-\epsilon_2=1$ (Theorem (3.3))
(*d*) $-\epsilon_1=\epsilon_2=1$ (Theorem (4.1))
(*e*) $-\epsilon_1=-\epsilon_2=1 \quad \mu>M^2$ (Theorem (4.2))
(*f*) $-\epsilon_1=-\epsilon_2=1 \quad \mu<M^2$ (Theorm (4.3))

will be termed principal conditions. The inequalities

(5.2)
(*a*) $\rho_2>\rho_4>\rho_3>\rho_1$ (Theorem (3.1))
(*b*) $\rho_2>\rho_1>\rho_3>\rho_4$ (Theorem (3.2))
(*c*) $\rho_3>\rho_1>\rho_2>\rho_4$ (Theorem (3.3))
(*d*) $\rho_4>\rho_2>\rho_1>\rho_3$ (Theorem (4.1))
(*e*) $\rho_4>\rho_3>\rho_1>\rho_2$ (Theorem (4.2))
(*f*) $\rho_1>\rho_3>\rho_4>\rho_2$ (Theorem (4.3))

will be termed principal inequalities,

Definition (5.2). *If the contracted curvature tensor* $\overset{\nu}{\underset{\lambda}{R}}$ *of* $g_{\lambda\mu}$ *satisfies the conditions of one of the Theorems* (3.1)—(4.3) *then* $g_{\lambda\mu}$ *will be called a "physical tensor" admitting the corresponding principal condition.* These definitions enable us to state in a concise form several Theorems.

Theorem (5.1). *A necessary and sufficient condition for a physical tensor* $g_{\lambda\mu}$ *to satisfy the conditions of theorem* (3.1) *is : The eigenvalue* ρ_4 *of its contracted curvature tensor is the second largest of* $\rho_1, \ldots . \rho_4$.

Proof. There is a one-to-one correspondence between Theorems (3.1)—(4.3) and the corresponding principal conditions (5.1) (*a*)—(*f*) as well as between the principal conditions (5.1) (*a*)—(*f*) and the corresponding principal inequalities. Hence there is a one-to-one correspondence between the Theorems (3.1)—(4.3) and the corresponding principal inequalities (5.2) (*a*)—(*f*).

Assume that the physical tensor $g_{\lambda\mu}$ satisfies the conditions of Theorem (3.1). Then it admits the principal condition (5.1)*a* and therefore also the principal inequality (5.2)*a*. Assume now that the given physical tensor has the property that the eigenvalue ρ_4 of its contracted curvature tensor is the second largest among $\rho_1, \ldots. \rho_4$. This is possible only for the principal inequality (5.2)*a*. Because the tensor $g_{\lambda\mu}$ is a physical one it must satisfy one of the Theorems (3.1)—(4.3) *i.e.*, Theorem (3.1).

Theorem (5.2). *A necessary and sufficient condition for a physical tensor* $g_{\lambda\mu}$ *to admit the conditions of one of the Theorems* (3.2) *and* (3.3) *is : The eigenvalue* ρ_4 *of its contracted curvature tensor is the smallest one of* $\rho_1, \ldots. \rho_4$

The proof follows the pattern of the previous proof.

Theorem (5.3). *In the case of Theorem* (5.2) *it is impossible to decide without further data whether the physical tensor* $g_{\lambda\mu}$ *satisfies the conditions of Theorem* (3.2) *or Theorem* (3.3)

Proof. Let us consider first the condition (3.8)*c*. Because in the case of Theorem (3.3) we have the principal condition (5.1)*c* it is irrelevant in this case whether $\mu > M^2$ or $\mu < M^2$. Neither one of these inequalities has any influence on the conditions of Theorem (3.3). Hence if it is not known which one of the inequalities $\mu \lessgtr M^2$ holds we do not have the counterpart of (3.8)*c* in Theorem (3.3). Consider now the first inequality (3.8)*b* which is formally the same as the first inequality (3.13)*b*. Comparing (5.2)*b* and (5.2)*c* we see that in both cases the eigenvalue ρ_1 is the second largest of $\rho_1, \ldots. \rho_4$. Denote by α, β the remaining two eigenvalues and consider the inequality

$$\alpha + \rho_1 < \beta + \rho_4 \tag{5.3}$$

It might be satisfied by

(*a*) $\alpha = \beta$. This case is excluded because of the characteristic [1111].
(*b*) $\alpha > \beta$. According to (5.2)*b*, *c* this case leads to (5.2)*b*
(*c*) $\alpha < \beta$. According to (5.2)*b*, *c* this case leads to (5 2)*c*.

Consider now the remaining inequalities (3.8)*b* (formally equivalent to the remaining inequalities (3.13)*b*)) which may be condensed into one

$$\rho_3 + \rho_4 \times 0 \tag{5.4}$$

where the symbol × denotes either one of the inequality symbols. The relation (5.4) could indicate which one of the Theorems (3.2), (3.3) is defined by Theorem (5.2) if the following conditions are satisfied. The relations

$$\alpha + \rho_4 \times 0$$
$$\beta + \rho_4 \times 0$$

are incompatible provided that in both cases the symbol × has the same interpretation. It is not difficult to see that these conditions are satisfied neither for $\alpha > \beta$ nor for $\alpha < \beta$. In this way we exhausted all possibilities which could have been indicative for our purposes.

The following two theorems will be proved simultaneously.

Theorem (5.4) *A necessary and sufficient condition for a physical tensor* $g_{\lambda\mu}$ *to admit the conditions of one of the Theorems* (4.1) *and* (4.2) *is : The eigenvalue* ρ_4 *of its contracted curvature tensor is the largest one of* $\rho_1, \ldots \rho_4$

Theorem (5.5). *In the case of Theorem* (5.3) *it is impossible to decide without further informations whether the physical tensor* $g_{\lambda\mu}$ *satisfies the conditions of Theorem* (4.1) *or Theorem* (4.2).

The proof of these two Theorems follows the pattern of the proofs of Theorems (5.2) and (5.3).

Theorm (5.6) *A necessary and sufficient condition that a physical tensor* $g_{\lambda\mu}$ *satisfies the conditions of Theorem* (4.3) *is : The eigenvalue* ρ_4 *of its contracted curvature tensor is the third largest among* $\rho_1, \ldots, \rho_4$.

The proof follows the pattern of the proof of Theorm (5.1)

Theorem (5.7) *A necessary and sufficient condition for a physical tensor* $g_{\lambda\mu}$ *to admit a* $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *mass is : The eigenvalue* ρ_4 *is either* $\left\{\begin{matrix}\textit{the second largest}\\ \textit{the largest}\end{matrix}\right.$ *or* $\left\{\begin{matrix}\textit{the smallest}\\ \textit{the third largest}\end{matrix}\right.$ *of* $\rho_1, \ldots, \rho_4$.

The proof follows by inspection of Theorems (5.1)—(5.6).

Remark (5.1). This finishes our considerations concerning the characteristic [1111]. It is not difficult to extend the present result to the cases of all *admissible* characteristics.

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NOTE ON THE DIRECT DETERMINATION OF STEADY-STATE THERMAL STRESSES IN CIRCULAR DISKS AND SPHERES

By

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Abstract. The object of this paper is to describe direct methods of finding stresses in circular disks and spheres under certain boundary conditions when there is steady heat flow in the material.

Introduction. Direct methods of solving some boundary-value problems of the theory of elasticity were discussed in previous papers by the author (Sen, 1946, 1957). It is shown in this note that the boundary-value problems of circular disks and spheres can be solved by the direct method even if there be steady temperature distribution in the medium. As illustrations, only two problems are considered. The first problem is concerned with the determination of displacements in a circular disk when these are prescribed on the boundary while the second problem discussed is that of a solid sphere having prescribed tractions on the surface. In either case it is supposed that the material is isotropic and there is steady flow of heat in it. It may be noted that direct methods for solving similar problems for other types of boundary conditions can also be easily deduced.

I. Two-dimensional problem of a circular disk

If the rectangular axes ox and oy are taken in the middle plane of a thin disk in a state of plane stress, we can assume $X_z=Y_z=Z_z=0$. T being the temperature at the point (x, y), stress-strain relations in usual notations are given by

$$\begin{aligned} e_{xx}-\alpha T&=\frac{1}{E}\left[X_x-\sigma Y_y\right],\\ e_{yy}-\alpha T&=\frac{1}{E}\left[Y_y-\sigma X_x\right],\\ e_{xy}&=\frac{2(1+\sigma)}{E}X_y, \end{aligned} \tag{1.1}$$

where E=Young's modulus, σ=Poisson's ratio, and α=Coefficient of linear expansion. From the above equations we get

$$\begin{aligned} \triangle_1&=e_{xx}+e_{yy}\\ &=\frac{1-\sigma}{E}\Theta_1+2\alpha T, \end{aligned} \tag{1.2}$$

where $\Theta_1 = X_x + Y_y$, and

$$\begin{aligned} X_x &= \frac{E}{1-\sigma^2}[\sigma\Delta_1 + (1-\sigma)\, e_{xx} - \alpha(1+\sigma)T], \\ Y_y &= \frac{E}{1-\sigma^2}[\sigma\Delta_1 + (1-\sigma)\, e_{yy} - \alpha(1+\sigma)T], \\ X_y &= \frac{E}{2(1+\sigma)}\, e_{xy}. \end{aligned} \tag{1.3}$$

Hence from the stress equations of equilibrium, we obtain

$$\begin{aligned} \nabla_1^2 u &= -\gamma \frac{\partial}{\partial x}[\Delta_1 - 2\alpha T], \\ \nabla_1^2 v &= -\gamma \frac{\partial}{\partial y}[\Delta_1 - 2\alpha T], \end{aligned} \tag{1.4}$$

in which $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; $\gamma = \frac{1+\sigma}{1-\sigma}$; u and v are components of displacement along the axes of x and y.

If there be steady flow of heat, we have

$$\nabla_1^2 T = 0. \tag{1.5}$$

Hence differentiating the equations (1.4) with respect to x and y, we get by adding the results

$$\nabla_1^2 \Delta_1 = 0. \tag{1.6}$$

T being a plane harmonic function with given values on the boundary, it can be uniquely determined. Let

$$T = \text{Re}\ \psi(z), \tag{1.7}$$

where $z = x + iy$ and Re denotes the 'real part.'

Suppose

$$\Delta_1 = \text{Re}\ \phi(z). \tag{1.8}$$

Then we can put

$$\begin{aligned} u &= \frac{\gamma}{4}\ \text{Re}\left[(a^2 - r^2)\, \frac{\phi(z) - 2\alpha\psi(z)}{z}\right] + \text{Re}g_1(z), \\ v &= \frac{\gamma}{4}\ \text{Re}\left[i(a^2 - r^2)\, \frac{\phi(z) - 2\alpha\psi(z)}{z}\right] + \text{Re}g_2(z) \end{aligned} \tag{1.9}$$

where $\text{Re}g_1(z)$ and $\text{Re}g_2(z)$ are the values of u and v on the boundary $r = a$, supposed to be known (Cf. Sen, 1957).

Since $\text{Re}\phi(z) = \Delta_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$,

we obtain from (1.9)
$$\phi(z) = \frac{2}{2+\gamma}[\alpha\gamma\psi(z) + g_1{}^1(z) + ig_2{}^1(z)]. \tag{1.10}$$

The value of $\phi(z)$ being determined in this way, expressions for u and v at any point can be found from (1.9)

2. Three-dimensional problem of a solid sphere

In the case of an isotropic elastic solid having steady temperature, we have by generalising the Beltrami-Michell equations, the stress equations of equilibrium as

$$
\begin{aligned}
&\nabla^2 X_x+\frac{1}{1+\sigma}\frac{\partial^2}{\partial x^2}\Big[\Theta+\alpha ET\Big]=0,\\
&\nabla^2 X_y+\frac{1}{1+\sigma}\frac{\partial^2}{\partial x\partial y}\Big[\Theta+\alpha ET\Big]=0,\\
&\nabla^2 X_z+\frac{1}{1+\sigma}\frac{\partial^2}{\partial x\partial z}\Big[\Theta+\alpha ET\Big]=0,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots
\end{aligned}
\tag{2.1}
$$

where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, $\Theta=X_x+Y_y+Z_z$,

α = coefficient of linear expansion, T=Temperature,

σ = Poisson's ratio, and E=Young's modulus, (Cf. Sharma, 1956). We have also in this case $\nabla^2\Theta=0$, and $\nabla^2 T=0$. (2.2)

Let us put $S=\Theta+\alpha ET$ (2.3)

so that we get $\nabla^2 S=0$. (2.4)

Multiplying the first of the equations (2.1) by x, the second by y and the third by z, we have by addition

$$x\nabla^2 X_x+y\nabla^2 X_y+z\nabla^2 X_z+\frac{1}{1+\sigma}\left[x\frac{\partial^2 S}{\partial x^2}+y\frac{\partial^2 S}{\partial x\partial y}+z\frac{\partial^2 S}{\partial x\partial z}\right]=0,$$

or

$$\nabla^2[xX_x+yX_y+zX_z]-2\left[\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right]+\frac{1}{1+\sigma}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right]\frac{\partial S}{\partial x}=0. \tag{2.5}$$

The second expression in the above equation vanishes on account of equilibrium.

Let X_r be the x—component of the traction on the spherical surface of radius r at the point (x, y, z).

Then

$$X_r=\frac{x}{r}X_x+\frac{y}{r}X_y+\frac{z}{r}X_z$$

Hence the equation (2.5) reduces to

$$\nabla^2(rX_r)+\frac{1}{1+\sigma}r\frac{\partial}{\partial r}\left(\frac{\partial S}{\partial x}\right)=0$$

or

$$\nabla^2(rX_r)+\frac{1}{1+\sigma}\frac{\partial}{\partial x}\left[r\frac{\partial S}{\partial r}-S\right]=0.$$

Similarly we shall have

$$\nabla^2(rY_r)+\frac{1}{1+\sigma}\frac{\partial}{\partial y}\left[r\frac{\partial S}{\partial r}-S\right]=0 \tag{2.6}$$

$$\nabla^2(rZ_r)+\frac{1}{1+\sigma}\frac{\partial}{\partial z}\left[r\frac{\partial S}{\partial r}-S\right]=0$$

Since S is a harmonic function, we may put

$$S=\Sigma S_n=\Sigma\Theta_n+\alpha E\Sigma T_n, \tag{2.7}$$

where S_n, Θ_n, T_n are solid homogeneous harmonics of nth degree.

Then

$$r\frac{\partial S}{\partial r}-S=\Sigma(n-1)S_n\,.$$

Remembering that

$$\nabla^2[(a^2-r^2)F_n]=-2(2n+3)F_n\,,$$

where F_n is a solid harmonic of nth degree, and a is a constant, we get,

$$\nabla^2\left[(a^2-r^2)\frac{\partial S_n}{\partial x}\right]=-2(2n+1)\frac{\partial S_n}{\partial x}\,.$$

For a sphere of radius a, the solutions of the equations (2.6) can therefore be written as

$$rX_r=\frac{1}{2(1+\sigma)}\sum\frac{n-1}{2n+1}(a^2-r^2)\frac{\partial S_n}{\partial x}+a\sum\frac{r^n}{a^n}X_n,$$

$$rY_r=\frac{1}{2(1+\sigma)}\sum\frac{n-1}{2n+1}(a^2-r^2)\frac{\partial S_n}{\partial y}+a\sum\frac{r^n}{a^n}Y_n \tag{2.8}$$

$$rZ_r=\frac{1}{2(1+\sigma)}\sum\frac{n-1}{2n+1}(a^2-r^2)\frac{\partial S_n}{\partial z}+a\sum\frac{r^n}{a^n}Z_n,$$

where X_n, Y_n, Z_n denote spherical surface harmonics of degree n such that

$$[X_r]_{r=a}\Sigma X_n,\ [Y_r]_{r=a}=\Sigma Y_n,\ [Z_r]_{r=a}=\Sigma Z_n \tag{2.9}$$

On account of equations of equilibrium, we have

$$\frac{\partial}{\partial x}\left(rX_r\right)+\frac{\partial}{\partial y}\left(rY_r\right)+\frac{\partial}{\partial z}\left(rZ_r\right)$$

$$=x\left[\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right]+y\left[\frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right]+z\left[\frac{\partial X_z}{\partial x}+\frac{\partial Y_z}{\partial y}+\frac{\partial Z_z}{\partial z}\right]$$

$$+X_x+Y_y+Z_z=\Theta$$

Hence differentiating the equations of (2.8) with respect to x, y and z, we get on adding

$$\Theta=\Sigma\Theta_n=-\frac{1}{1+\sigma}\sum\frac{n-1}{2n+1}\left[x\frac{\partial S_n}{\partial x}+y\frac{\partial S_n}{\partial y}+z\frac{\partial S_x}{\partial z}\right]$$

$$+a\sum\left[\frac{\partial}{\partial x}\left(\frac{r^n}{a^n}X_n\right)+\frac{\partial}{\partial y}\left(\frac{r^n}{a^n}Y_n\right)+\frac{\partial}{\partial z}\left(\frac{r^n}{a^n}Z_n\right)\right]$$

or

$$\Sigma S_n - \alpha E \Sigma T_n = -\frac{1}{1+\sigma}\sum \frac{(n-1)n}{2n+1} S_n + a\Sigma \Psi_n \tag{2.10}$$

in which

$$\Psi_n = \frac{\partial}{\partial x}\left[\frac{r^{n+1}}{a^{n+1}} X_{n+1}\right] + \frac{\partial}{\partial y}\left[\frac{r^{n+1}}{a^{n+1}} Y_{n+1}\right] + \frac{\partial}{\partial z}\left[\frac{r^{n+1}}{a^{n+1}} Z_{n+1}\right] \tag{2.11}$$

The result (2.10) gives

$$S_n = \frac{(1+\sigma)(2n+1)}{(1+\sigma)(2n+1)+n(n-1)}\left[a\Psi_n + \alpha E T_n\right] \tag{2.12}$$

If the applied surface tractions be given, Ψ_n will be known. T_n will be determined from the solution of the heat conduction equation $\nabla^2 T=0$ with given distribution of temperature on the surface of the sphere. Hence from (2.12) S_n will be known. S_n being thus determined, X_r, Y_r, Z_r can be obtained from the equations (2.8).

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TRANSITION PHENOMENON IN PHYSICAL PROBLEMS

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1. Introduction. There is hardly any branch of science or technology in which we do not come across transition from one state into another. Elasticity, plasticity, visco-elastic deformations, boundary layers, shocks, creep, fatigue, relaxation, oscillations and adiabatic theory in Quantum Mechanics are well-known examples. At present, they are generally treated as discontinuities. For elastic-plastic deformations, the ideal model has to assume some type of yield condition. For creep, not only the yield condition, but also a number of creep-strain laws have to be assumed. For boundary layers, the pressure from the irrotational flow has to be used in the layer. In shocks, a number of jump conditions have to be taken. A large amount of work has been done on these problems by Stokes, Poincaré, Horn, Birkhoff, Noaillon, Tamarkin, Perron, Trijitzinski, Twittin, Hukuhara, Jeffreys, Langer, Wasow, Levinson, Bellman, Stoker, Reissner, Lin and Friedrichs. [1]

2. Transition concept. The concept of a transition state between two adjacent states of a medium has not attracted much attention due to the non-linearity involved in the analytical treatment. In one of his memoirs, Saint-Venant [2] refers to the work of Tresca wherein he assumes a mid-state in which elasticity alters to plasticity or as Tresca terms it, fluidity. The following remarks in Todhunter and Pearson's 'History of elasticity and strength of materials' [3] are interesting :

> "Saint-Venant distinguishes in his cylinder only two zones, an elastic and a plastic one, but Tresca supposes a mid-zone......Saint-Venant's discussion has the theoretical advantage, but it seems not impossible that physically something corresponding to Tresca's mid-zone has an existence."

A recent numerical study (4) on flow and deformation theories in plasticity was undertaken to see to what extent a continuous approximation, which involves the idea of transition, to an elastic-plastic material in terms of the stress-stain law, would lead to a satisfactory convergent solution. The results obtained showed excellent approximation and convergence to the elastic-perfectly plastic solution.

3. Measure of deformation. Irreversibility is involved in all transition problems and workers for over two centuries have treated such problems as linear, with the result, that the discontinuities, singular surfaces, non-differentiable regions had to be introduced over which two successive states of a medium were matched together with the help of semi-empirical laws. Mathematicians tackled the corresponding problems and their solutions helped to increase the semi-empirical laws to be assumed. This legacy has also permeated the present trend to explain experimental results relating to a large number of problems in 'Continuum Mechanics.' In general, the assumptions made are—

(1) Complex constitutive equations ;
(2) Semi-empirical conditions.

The formal frame-works so constructed have not been all successful and the best of theories have not fulfilled our hopes. The formalism and semi-empiricism adopted in such problems have become more and more complicated without evolving any simple concept governing them. One source of all these troubles is the use of classical measures of deformations produced in a medium, even when we know from experiment that non-linearity is a characteristic of such deformed media. The dynamical equations of equilibrium and the concept of stresses are well-defined; only the measures of deformation are flexible, as it should be. Any restriction placed on these measures strains the constitutive equations unnecessarily into complex forms, even in cases where we know it should not be done. Plastic and creep effects in metals are well-known examples. In such cases, the strain is small and no refinement of constitutive equations can better the results. Such problems should be treated as transition phenomena. It, therefore, appears that we should construct generalized measures to bring the full impact of modern measure theory to our aid. This is readily suggested by problems relating to high speed structures, and to those in creep, relaxation and fatigue where even the popular Hencky's logarithmic measure only gives a constant value of the rate of creep-strain. The present state of visco-elastic and rheological problems and the shock structure problems also indicate that instead of increasing incessantly the complexity of the constitutive equations, attempts should be made to evolve generalized measures. The function of such measures should be to condense the non-linear effects of the deformation and to show that no semi-empirical laws are necessary.

4. Analytical and physical identification of transition state.

The second concept involved in the treatment of transition phenomenon is the identification of the transition states with the turning points of the differentiable system describing the medium. In general, these turning points will correspond to infinite contractions or extensions of the medium. The metric or fundamental tensors of the space may therefore be used, and the physical transition may be identified with the limiting cases when the axis of the deformation-ellipsoid tend to either zero or infinite. To illustrate in a simple manner some of these ideas we may discuss the transition of a compressible flow from supersonic to subsonic. Some of the assumptions that are made by neglecting viscosity come out without assuming any jump conditions.

5. One-dimensional steady compressible flow.

In one-dimensional steady viscous compressible flow the equations of motion are

$$\rho u = m,$$

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \frac{4}{3}\frac{\partial}{\partial x}\left(\mu \frac{\partial}{\partial x}\right) \tag{5.1}$$

Assuming $\rho=f(p)$, and putting $P=\partial u/\partial x$, we get

$$mP\left(1-\frac{c^2}{u^2}\right)=\frac{4}{3}P\frac{d}{du}(\mu P), \tag{5.2}$$

where $c^2=dp/d\rho=$the local sound velocity.

Since $P\neq 0$, as otherwise $u=0$, we have, if μ is taken as constant

$$\frac{dP}{du}=\frac{3m}{4\mu}\left(1-\frac{c^2}{u^2}\right) \tag{5.3}$$

The three singular points, which can be interpreted as the transition points, are

(i) $\mu\to 0$, (ii) $\mu\to\infty$ (iii) $u\to\pm c$

The first, $\mu\to 0$, ($R\to\infty$, R being the Reynolds number) corresponds to the case when the fluid tends to become non-viscous at the edge of a boundary layer. Over this transition the pressure-density relation is

$$p=k-\frac{m}{\rho},$$

which may be expected to hold good for gases with small viscosity.

The second, $\mu\to\infty$ ($R\to 0$) corresponds to the slow motion of highly viscous liquids.

The third, $u\to\pm c$, indicates the transition from supersonic to subsonic over the transition when the Mach number is 1. It is significant that μ has not been taken to be zero and no particular specifying condition in the form $p=f(\rho)$ has been taken. Over the shock,

$$\frac{\partial p}{\partial \rho}\sim u\sim m/\rho,$$

and hence $p\sim m\log(\rho/k)$, a relation which can be expected to be true for weak shocks where the entropy change can be neglected.

From (5.3) we see that if $\mu\to 0$ and $u\to\pm c$ simultaneously, dP/du is indeterminate. This shows that viscosity is significant over shock-transition region and should not be taken equal to zero.

Again, over transition $dP/du\to 0$. Putting this condition in (5.1) we get

$$m\frac{\partial u}{\partial x}=-\frac{\partial p}{\partial x},$$

$$\text{or,}\quad m(u_2-u_1)=p_1-p_2, \tag{5.4}$$

which is the momentum equation. This has been obtained without taking $\mu=0$, as is done in current literature.

In like manner for two-dimensional symmetrical flow we get

$$2\pi\rho r u=m,\quad ru=U,\quad \rho=m/2\pi U$$

$$\rho u\frac{\partial u}{\partial r}=-\frac{\partial p}{\partial r}+\frac{4}{3}\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)\right] \tag{5.5}$$

These give

$$\frac{m}{2\pi r}\left(\frac{1}{r}\frac{\partial U}{\partial r}-\frac{U}{r^2}\right)=\frac{c^2m}{2\pi U^2}\frac{\partial U}{\partial r}+\frac{4}{3}\mu\frac{\partial U}{\partial r}\frac{\partial}{\partial U}\left(\frac{\mu}{r}\frac{\partial U}{\partial r}\right),$$

which can be re-written as

$$\frac{d}{dU}\left(\int\frac{U}{r^3}\,dr+\frac{8\pi\mu}{3mr}\frac{dU}{dr}\right)=\frac{1}{r^2}-\frac{c^2}{U^2}. \tag{5. 6}$$

Once again we get transition at $u\rightarrow\pm c$, $(M\rightarrow\pm 1)$.

Similarly we may construct more difficult cases. For example, for the one-dimensional flow of an ideal gas

$$p=gR\rho T,\ \frac{\partial p}{\partial\rho}=gRT=c^2, \tag{5.7}$$

$$\frac{4}{3}\frac{\mu}{m}\frac{\partial u}{\partial x}=u+\frac{gRT}{u}-c_1=u+\frac{c^2}{u}-c_1,$$

$$\frac{k}{m}\frac{\partial T}{\partial x}=c_1u-c_2-\tfrac{1}{2}u^2+\frac{gR}{\Upsilon-1}T$$

$$=c_1u-c_2-\tfrac{1}{2}u^2+\frac{c^2}{\Upsilon-1}. \tag{5.8}$$

These give

$$\frac{4}{3}\frac{\mu}{m}P\frac{\partial p}{\partial u}=P-\frac{gRT}{u^2}P+\frac{gR}{u}\frac{\partial T}{\partial u}P,$$

or, $$\frac{4}{3}\frac{\mu}{m}\frac{\partial p}{\partial u}=1-\frac{c^2}{u^2}+\frac{gR}{up}\frac{\partial T}{\partial x}.$$

$$=1-\frac{c^2}{u^2}+\frac{mgR}{uPk}\left(c_1u-c_2-\tfrac{1}{2}u^2+\frac{c^2}{\Upsilon-1}\right), \tag{5.9}$$

c_1, c_2 being constants of integration.

This may be re-written as

$$(u^2-c^2)\left(\frac{pk}{u}+\frac{mgR}{\Upsilon-1}\right)$$

$$=\frac{1}{3}\frac{u}{m}\frac{d}{du}\left[2\mu kp^2-3m^2gR\left(c_1u-c_2\log u-\frac{1}{4}u^2-\tfrac{1}{2}\frac{u^2}{\Upsilon-1}\right)\right],$$

which may be written as

$$\frac{1}{3m}\frac{dQ}{du}=\left(1-\frac{c^2}{u^2}\right)\left(Pk+\frac{mgRu}{\Upsilon-1}\right) \tag{5.10}$$

which again shows transition at $M=\pm 1$,

6. Generalized strain measures

The use of generalized strain measures can best be illustrated by taking problem in elasticity, plasticity, creep and fatigue [5].

As the strain tensor e_{ij} is symmetric we may associate it with a quadric surface, known as Cauchy's strain quadric, at any point P, such that

$$e_{ij}x^i x^j = \text{a constant.} \tag{6.1}$$

If Λ_i be its principal axes and a_i and x_i are the coordinates before and after strain we may write

$$\Sigma da_i^2 = \Sigma dx_i^2 / \Lambda_i^2, \tag{6.2}$$

Any suitable functions of Λ_i which vanishes for $\Lambda_i = 1$ when there is no deformation may constitute a measure of strain. If we put

$$\Lambda = (1-ne)^{-1/n}, \tag{6.3}$$

we notice that it includes all the known measures. In fact we have the following results :

$$\begin{array}{lll} n = -1, & = 1+e, & \text{Cauchy (C)} \\ n = -2, & = (1+2e)^{1/2}, & \text{Green (G)} \\ n = 0, & = \exp(e), & \text{Hencky (H)} \\ n = 2, & = (1-2e)^{1/2}, & \text{Almansi (A)} \\ n = 1, & = (1-e)^{-1}, & \text{Swainger (S)} \\ n = \infty, & = 1, & \text{No strain (N)} \end{array} \tag{6.4}$$

We can therefore call n the coefficient of strain measure and Λ as a generalized strain measure. The strain components of the measures in (6.4) have been calculated by Cauchy, Green[6], Murnaghan [7], Reiner and Hanin [8] and others. Seth [9] has stressed the use of the A-measure in a number of applications.

It has been shown that the use of the Almansi's measure solves problems in elastic-plastic deformation without using any yield condition. In fact it gives a number of other interesting results, like the following [10] :

(i) A distinction is made between transition due to extension and that due to contraction.

(ii) It shows work-hardening when it occurs.

(iii) In torsion it shows that the axis of the cylinder never becomes plastic.

and (iv) In plastic sheet bending it predicts the form of the plate when it becomes plastic.

These cannot be given by classical elastic-plastic theories. For creep, relaxation and fatigue, the measure in (6.3) may be generalized to

$$\Lambda = \left(1 - ne^{1/m}\right)^{-1/n} \tag{6.5}$$

where m is found to be the index of transient creep. This can be shown to give all the known creep-strain laws, whose assumption then becomes unnecessary [11]. For example, for the one-dimensional case the well-known Norton's law can be deduced in the following manner :

Let A be the cross-section of a rod of length l and A_0, l_0 the initial values of these quantities. Further, let a be the average constant strain rate, so that after a time t the total ductile strain is $a\,t$. From (6.5) we have another expression for the total strain. Equating them we get

$$at=\frac{1}{n}\left[1-\left(\frac{l_0}{l}\right)^n\right], \qquad m=1. \tag{6.6}$$

The condition of incompressibility gives

$$Al=A_0l_0. \tag{6.7}$$

If P be the total load we have also the relation $P=T_yA=T_0A_0$. (6.8)

Differentiating (6.6) with respect to t and combining the result with (6.7) and (6.8) we get

$$\Upsilon=\text{Rate of creep strain}=\frac{1}{l}\frac{dl}{dt}=a\left(\frac{T_y}{T_0}\right)^n$$

which is Norton's law.

In like manner the well-known Andrade's law which holds good for a variety of metals at different temperatures may be deduced from (6.5) by adding together the two creep strain rates obtained by putting $m=1$, $n=0$ and $m=3$, $n=1$ in (6.5). For a wire of initial and final lengths l, l_0 respectively it takes the form

$$l=l_0(1+\beta t^{1/3})e^{kt}.$$

Laws due to Odqvist and Kachanov and others may be similarly deduced.

Problems due to fatigue and initial strain can be treated on similar lines.

7. Rheological problems

By altering the classical strain-velocity measure

$$2e_{ij}=\dot{u}_{i,j}+\dot{u}_{j,i}$$

a number of problems relating to visco-elastic fluids may be treated. For principal components we may put the generalized strain-velocity measure in the form

$$\epsilon_{ii}=\frac{k}{n^pm^p}\left[1-\left(1-me_{ii}\right)^n\right]^p, \tag{7.1}$$

m and k help to correct the dimensions, n, p are the measure-indices. The generalized strain-velocity tensor can be readily constructed from (7.1) [12].

For $n=1$, $p=1$, $\epsilon_{ii}=e_{ii}$, which is the ordinary measure

For $n=1$, $\epsilon_{ii}=e_{ii}^p$, which pertains to power-law fluids

For $n\to 0$, $p=1$, $\epsilon_{ii}=-\frac{1}{m}\log(1-me_{ii})$ which is the logarithmic measure.

It may therefore be concluded that the concept of the asymptotic solution at the turning points of a differential system combined with that of the generalized measure for a deformed medium can give solution of a number of transition problems, without assuming semi-empirical laws whose number seems to be on the increase.

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ON THE THEORY OF AREAL SPACES

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1. As is well-known, a Finsler space F_n is an n-dimensional differentiable manifold with the measure of its line element (x, dx) defined by a homogeneous function $F(x, dx)$ of degree one in dx, *i.e.*, the curve length s of a curve $x=x(t)$ in F_n is given by

$$s=\int F\left(x, \frac{dx}{dt}\right) dt, \qquad (t : \text{parameter of the curve})$$

where we abbreviate, for the sake of simplicity, the superscript of n coordinates x of a point or dx^i of a tangent vector and denote it only by a letter x or dx. The function $F(x, dx)$ is called the *fundamental function of* F_n. Obviously a Riemannian space is a special case of a Finsler space, *i.e.*,

$$F(x, dx)=\sqrt{g_{ij}(x)dx^i\, dx^j}.$$

Repeated indices imply summation throughout this paper. Although the consistency of this space was suggested already by Riemann in 1864, the geometry in this space was studied first by P. Finsler in 1918 and afterwards by L. Berwald, E. Cartan and many others (see H. Rund ; Differential geometry of Finsler spaces, Springer 1959).

On the other hand, E. Cartan introduced a so-called Cartan space in 1933 where the measure of its hyperplane element (x, p) is given *a priori* by a function $F(x, p)$ homogeneous of degree one in every line element contained in the element (x, p), *i.e.*, the area of a domain on a hypersurface

$$x^i=x^i(u^1, u^2, \ldots, u^{n-1}), \qquad i=1, 2, \ldots, n,$$

is given by

$$S=\underbrace{\int \cdots \int}_{(n-1)} F\left(x, \frac{\partial x}{\partial u}\right) du^1 du^2 \ldots du^{n-1}.$$

This space is also a generalization of Riemannian space. In the last space the fundamental function reduces to

$$F\left(x, \frac{\partial x}{\partial u}\right)=\sqrt{\det. \left| g_{ij}(x)\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}\right|}, \qquad \alpha, \beta=1, 2, \ldots, n-1.$$

These facts give rise to the problem : To discuss the geometry in the space, called *areal space* and denoted by $\overset{(m)}{A_n}$, with the measure of its m-dimensional plane element $(x, \underset{(m)}{p})$ defined by a function $F(x, \underset{(m)}{p})$ homogeneous of degree one in every line element contained in the element $(x, \underset{(m)}{p})$, *i.e.*, the area of a domain on an m-dimensional subspace

$$x^i = x^i(u^1, u^2, \ldots, u^m)$$

is given by

$$\underset{(m)}{S} = \underset{(m)}{\int \cdots \int} F\left(x, \frac{\partial x}{\partial u}\right) du^1 du^2 \ldots du^m,$$

where $1 \leqq m \leqq n-1$.

It is evident that this space is also a generalization of Riemannian space, *i.e.*, for Riemannian space F reduces to

$$F\left(x, \frac{\partial x}{\partial u}\right) = \sqrt{\det.\left|g_{\alpha\beta}\right|},$$

$$g_{\alpha\beta} \overset{\mathrm{def}}{=} g_{ij}(x) \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}, \qquad \alpha, \beta = 1, 2, \ldots, m.$$

Not only Riemannian space but also Finsler and Cartan spaces are particular cases of this areal space, *i.e.*, $A_n^{(1)}$ and $A_n^{(n-1)}$ for $m=1$ and $m=n-1$ respectively.

This space was taken up first in 1936 by the present author and then was treated by many others in the last 17 years. The theory of Areal Spaces is not so easy to discuss because we can not treat it in the same way as those of Riemannian, Finsler and Cartan spaces due to the difficulty which occurs only for the areal space.

In the present paper the theory established by my school members and others is explained only in its outline, not in detail.

2. Let M_n be a differentiable manifold of dimension n with local coordinate neighborhood U of which a point P has coordinates x^i $(i=1, 2, \ldots\ldots, n)$, and consider the space $\tilde{M}_n^{[m]}$ of m-frames on M_n, where a frame p consists of m linearly independent vectors p_α^i at a point P $(\alpha=1, 2, \ldots, m)$.[1] $\tilde{M}_n^{[m]}$ is a fibre bundle over M_n with the projection γ of $\tilde{M}_n^{[m]}$ onto M_n and is acted on by the group Ψ of fibre preserving diffeomorphisms $p \to p\psi$: $p_\alpha^i \to p_\beta^i \psi_\alpha^\beta$, where ψ_α^β is any non-singular matrix with positive determinant. The quotient space $M_n^{[m]} = \tilde{M}_n^{[m]}/\Psi$ is the space of oriented contact elements of M_n of first order and rank m, and is also a fibre bundle over M_n with the projection $\pi : M_n^{[m]} \to M_n$.

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[1] If we consider an m-dimensional subspace M_m imbedded in M_n and M_m is expressed in the parametric forms $x^i = x^i(u^1, u^2, \ldots, u^m)$, then we may take $\frac{\partial x^i}{\partial u^\alpha}$ in place of p_α^i.

An areal metric or structure of dimension m on M_n is defined by a positive differentiable function F on $\tilde{M}_n^{[m]}$ with the property

$$F(p\psi) = |\psi| F(p), \tag{1}$$

where $|\psi|$ denotes the determinant of the matrix $\psi(=((\psi^\alpha_\beta)))$. An areal space $A_n^{(m)}$ is an $\tilde{M}_n^{[m]}$ with an areal metric or structure of dimension m.

The changes of local coordinates on $\tilde{M}_n^{[m]}$ are of the form

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \cdots\cdots, x^n), \qquad \bar{p}^i_\alpha = \frac{\partial \bar{x}^i}{\partial x^j} p^j_\alpha \tag{2}$$

and F becomes a function $F(x^i, p_\alpha^i)$ in terms of the local coordinates x^i, p_α^i.

We introduce the following notations :

$$F\Big|^\alpha_h \overset{\text{def}}{=} \frac{\partial F}{\partial p^h_\alpha}$$

$$p^\alpha_i \overset{\text{def}}{=} \frac{1}{F} F\Big|^\alpha_i = (\log F)\Big|^\alpha_i, \tag{3}$$

$$L^{\alpha\beta}_{ij} \overset{\text{def}}{=} (\log F)\Big|^\alpha_i\Big|^\beta_j + p^\alpha_i p^\beta_j = p^\alpha_i\Big|^\beta_j + p^\beta_i p^\alpha_j \tag{4}$$

$$= \frac{1}{F} \frac{\partial^2 F}{\partial p^i_\alpha \, \partial p^j_\beta} - \frac{1}{F^2} \frac{\partial F}{\partial p^i_\alpha} \frac{\partial F}{\partial p^j_\beta} + \frac{1}{F^2} \frac{\partial F}{\partial p_\beta} \frac{\partial F}{\partial p^j_\alpha}, \qquad \text{(Legendre form)}$$

$$E_i \overset{\text{def}}{=} \frac{1}{F}\left\{\frac{\partial F|^\alpha_i}{\partial u^\alpha} - \frac{\partial F}{\partial x^i}\right\} \quad \text{for } x^i = x^i(u^\alpha), \qquad \text{(Euler vector)} \tag{5}$$

$$p^i_j \overset{\text{def}}{=} p^i_\alpha p^\alpha_j, \quad q^i_j \overset{\text{def}}{=} \delta^i_j - p^i_j. \tag{6}$$

Then evidently p^α_i, $L^{\alpha\beta}_{ij}$, E_i, p^i_j and q^i_j have vector or tensor properties with respect to the indices i and j as follows from (2), and for the diffeomorphism $p \to p\psi$

$$p^\alpha_i \to p^\beta_i \psi^{*\alpha}_{\ \beta}, \qquad L^{\alpha\beta}_{ij} \to L^{\gamma\delta}_{ij} \psi^{*\alpha}_{\ \gamma} \psi^{*\beta}_{\ \delta}, \tag{7}$$

where $(\psi^{*\alpha}_{\ \beta})$ is the conjugate matrix of (ψ^α_β) :

$$\psi^{*\alpha}_{\ \beta} \psi^\beta_\gamma = \psi^\alpha_\beta \psi^{*\beta}_{\ \gamma} = \delta^\alpha_\gamma. \tag{8}$$

The following identities are found easily :

$$p^\alpha_i p^i_\beta = \delta^\alpha_\beta, \tag{9}$$

$$L^{\alpha\beta}_{ij} p^i_\gamma = 0, \qquad L^{\alpha\beta}_{ij} p^j_\gamma = 0. \tag{10}$$

It is shown that the Legendre form $L^{\alpha\beta}_{ij}$ splits to two factors for a Riemannian space as well as for Finsler and Cartan spaces such as

$$L^{\alpha\beta}_{ij} = g''_{ij} g^{\alpha\beta}, \tag{11}$$

where $g^{\alpha\beta}$ are contravariant components of the tensor

$$g_{\alpha\beta} \overset{\text{def}}{=} g_{ij}\, p^i_\alpha\, p^j_\beta \tag{12}$$

and

$$g''_{ij} \overset{\text{def}}{=} g_{ij} - g'_{ij}\,, \qquad g'_{ij} \overset{\text{def}}{=} g_{\alpha\beta}\, p^\alpha_i\, p^\beta_j\,. \tag{13}$$

However, in a general areal space the form $L^{\alpha\beta}_{ij}$ does not split.

We can introduce the following tensor with $2m$ indices :

$$g_{i_1\cdots i_m, j_1\cdots j_m} \overset{\text{def}}{=} \sum_{a=0}^{m} \binom{m}{a}^2 \epsilon_{\alpha_1\cdots\alpha_m} \epsilon_{\beta_1\cdots\beta_m} F^2 L^{\alpha_1\ \beta_1}_{[i_1\ [j_1} L^{\alpha_2\ \beta_2}_{[i_2\ [j_2} \cdots\cdots L^{\alpha_a\ \beta_a}_{i_a\ j_a} \tag{14}$$

$$\times \left(p^{\alpha_{a+1}}_{i_{a+1}} p^{\beta_{a+1}}_{j_{a+1}}\right) \left(p^{\alpha_{a+2}}_{i_{a+2}} p^{\beta_{a+2}}_{j_{a+2}}\right) \cdots\cdots \left(p^{\alpha_m}_{i_m]} p^{\beta_m}_{j_m]}\right),$$

where $\epsilon_{\alpha_1 \ldots\ldots \alpha_m}$ is Eddington's epsilon and brackets mean the skew-symmetric parts with respect to $i_1, i_2 \ldots\ldots i_m$ and $j_1, j_2 \ldots\ldots j_m$ respectively.

This tensor has the following properties :

(a) skew-symmetric with respect to m subscripts i's as well as to m subscripts j's and symmetric with respect to two systems of these m subscripts, *i.e.*,

$$g_{i_1 \cdots i_m, j_1 \cdots j_m} = g_{j_1 \cdots j_m, i_1 \cdots i_m},$$

(b) invariant under the group Ψ, *i.e.*, the functions $g_{i_1, \cdots i_m\, j_1, \cdots j_m}(x,p)$ are homogeneous of degree zero with respect to p^i_α for any fixed value of α, *i. e.*,

$$p^k_\beta \frac{\partial}{\partial p^k_\alpha} g_{i_1 \cdots i_m, j_1 \cdots j_m} = 0, \tag{15}$$

for any values of α and β.

(c) the relations hold good

$$g_{i_1 \cdots i_m, j_1 \cdots j_m}\, p^{j_1 \cdots j_m} = m\,!\, G_{i_1 \cdots i_m}\,, \tag{16}$$

$$g_{i_1 \cdots i_m, j_1 \cdots j_m}\, p^{i_1 \cdots i_m} p^{j_1 \cdots j_m} = (m\,!)^2 F^2\,, \tag{17}$$

$$p^{i_1 \cdots i_m} \frac{\partial}{\partial p^k_\alpha} g_{i_1 \cdots i_m, j_1 \cdots j_m} = 0, \tag{18}$$

where

$$p^{i_1\cdots i_m} \overset{\text{def}}{=} m!\, p^{[i_1}_{1}\, p^{i_2}_{2} \ldots p^{i_m]}_{m},$$

$$G_{i_1\cdots i_m} \overset{\text{def}}{=} m!\, F^2\, p^{1}_{[i_1}\, p^{2}_{i_2} \cdots p^{m}_{i_m]}\,.$$

By means of this tensor which we shall name the *metric m-tensor* in the areal space under consideration, we can define the measure of a contravariant m-vector $X^{i_1\cdots i_m}$ by

$$X^2 = \frac{1}{(m!)^2}\, g_{i_1\cdots i_m,\ j_1\cdots j_m}\, X^{i_1\cdots i_m}\, X^{j_1\cdots j_m}$$

and also that of a covariant m-vector in the similar way as usual. For Riemannian, Finsler and Cartan spaces, this tensor splits also into the m-rowed determinants constructed of the components of the metric tensor g_{ij}, *i.e.*

$$g_{i_1\cdots i_m,\ j_1\cdots j_m} = m!\, g_{[i_1\,[j_1} g_{i_2 j_2} \cdots g_{i_m]\,j_m]}, \tag{19}$$

which can be proved on account of the relation (11). But this fact is not true in general. Hence at first we have to study the special class of areal spaces for which there exists a tensor g_{ij} and this relation holds good. Such a space is named an *areal space of the metric class.* Then we prove, on making use of (14) and (18),

Theorem 1. *For an areal space of the metric class the Legendre form splits into* $g''_{ij}\, g^{\alpha\beta}$ *and the metric tensor* g_{ij} *can be defined uniquely by*

$$g_{ij} = g_{\alpha\beta}\, p^{\alpha}_{i}\, p^{\beta}_{j} + g''_{ij}$$

provided det. $\left| g^{\alpha\beta} \right| \neq 0.$

Theorem 2. *The necessary and sufficient condition that the areal space be of the metric class is that there exists a non-singular symmetric tensor* g_{ij} *which satisfies the following three equations :*

(i) det. $\left| g_{\alpha\beta} \right| = F^2,$

(ii) $p^{\alpha}_{i} = g_{ij}\, g^{\alpha\beta}\, p^{j}_{\beta},$ (*transversal relation*)

(iii) $g_{ij}\Big|^{\gamma}_{s}\, p^{i}_{h}\, q^{j}_{k} = 0,$

where

$$g_{ij}\Big|^{\gamma}_{s} = \frac{\partial g_{ij}}{\partial p^{s}_{\gamma}}\,.$$

If the relations (i) and

(iii′) $g_{ij}\Big|^{\gamma}_{s}\, p^{j}_{h} = 0$

hold good, then the space is of the metric class, because from (i) and (iii′), we can derive easily (ii) and (iii).

Theorem 2 leads us to ascertain the remarkable and final result established by T. Tándai :

Theorem 3. *An areal space of the metric class must be one of Riemannian, Finsler and Cartan spaces.*

This theorem shows us that the discussion of the space of the metric class is not needed in the theory of areal spaces.

3. The next step is to study the areal space of larger class than the metric class, because the Legendre form does not split in general.

If we can find a symmetric tensor $g_{ij}(x,p)$ of the second order, satisfying the following three conditions, it will be fairly convenient to study the space, since we may proceed in the same way as in the case of Riemannian or Finsler space.

(A) g_{ij} are constructed algebraically of the function $F(x, p)$ and its derivatives of first and second orders with respect to p's.

(B) g_{ij} are invariant under the group Ψ, *i.e.*,

$$g_{ij}(x, p\,\psi)=g_{ij}(x, p), \quad \psi \in \Psi,$$

(C) det. $\left| g_{\alpha\beta} \right| = F^2$, provided $g_{\alpha\beta}=g_{ij}\, p^i_\alpha\, p^j_\beta$.

However, unfortunately, such a tensor does not exist in general. Accordingly, we shall consider now such a class of areal spaces in which we can find a symmetric tensor $g_{ij}(x,p)$ satisfying the conditions (A) and (B). Such a space is called an *areal space of the submetric class.* In this space we can modify $g_{ij}(x, p)$ such that (C) is satisfied, *i.e.*, we take ρg_{ij} in place of g_{ij} as $\rho^m \overset{\text{def}}{=} F^2/\,\text{det.}\left|g_{\alpha\beta}\right|$ is an invariant under Ψ.

In fact there are examples of such spaces as the following theorem shows.

Theorem 4. *If the integers m and n are relatively prime, then a regular areal space $A_n^{(m)}$ is always of the submetric class.*

An areal space is called *regular*, when the $\binom{n}{m}$-rowed determinant $|g_{i[m]j[m]}|$ is different from zero.

For example, when $n=km+1$, we construct

$$G^{i_1 j_1} = \epsilon^{i[n]}\, \epsilon^{j[n]}\, g_{i_1\ldots i_m, j_1\ldots j_m}\, g_{i_{m+1}\ldots i_{2m}, j_{m+1}\ldots j_{2m}} \cdots$$
$$\times g_{i_{(k-1)m+1}\ldots i_{km}, j_{(k-1)m+1}\ldots j_{km}}$$

and modifying on making use of det.$|\,G^{ij}\,|$, we obtain the required g_{ij}.

For the sake of brevity, we use the notation

$$(20)\quad \begin{cases} \epsilon^{j[n]} \overset{\text{def}}{=} \epsilon^{j_1 j_2 \ldots j_n}, \quad \text{(Eddington's epsilon)} \\ p^{i[m]} \overset{\text{def}}{=} m!\, p_1^{[i_1}\, p_2^{i_2} \cdots p_m^{i_m]}, \\ g_{i[m]j[m]} \overset{\text{def}}{=} g_{i_1\ldots i_m, j_1\ldots j_m}, \end{cases}$$

and so on, which are skew-symmetric with respect to the superscripts or subscripts i's or j's respectively.

Let us now consider a regular areal space of the submetric class. Then there exists a symmetric tensor g_{ij} which is named the metric tensor and we assume det. $\left| g_{ij} \right| \neq 0$.

Put

$$g_{\alpha\beta} = \rho g_{ij}\, p^i_\alpha\, p^j_\beta$$

and determine ρ such that

$$\text{det.} \left| g_{\alpha\beta} \right| = F^2,$$

which is possible if det. $\left| g_{ij}\, p^i_\alpha p^j_\beta \right| \neq 0$. Then we name

$$\overset{*}{g}_{ij} \overset{\text{def}}{=} \left(p^\alpha_i\, p^\beta_j + \frac{1}{m} L^{\alpha\beta}_{ij} \right) g_{\alpha\beta} \tag{21}$$

the *normalized metric tensor.*

Theorem 5. *The normalized metric tensor satisfies the relations* :

$$\overset{*}{g}_{ij}\, p^i_\alpha\, p^j_\beta = g_{\alpha\beta}, \quad p^\alpha_i = \overset{*}{g}_{ij}\, g^{\alpha\beta}\, p^j_\beta, \tag{22}$$

$$C^{\ \ \alpha}_{ij\alpha} = 0, \tag{23}$$

where $\qquad C^{\ \ \alpha}_{ij\beta} \overset{\text{def}}{=} q^h_i\, p^k_\beta\, \overset{*}{g}_{hk} \Big|^{\alpha}_{j}.$

We have to remember that (22) and $C^{\ \ \alpha}_{ij\beta} = 0$ are the necessary conditions for the space of the metric class (Theorem 2). The condition (23) is weaker than $C^{\ \ \alpha}_{ij\beta} = 0$. The necessary and sufficient condition that an areal space of the submetric class be of the metric class is obtained in many different ways, for example,

(a) $\quad C^{\ \ \alpha}_{ij\beta} = 0$ (Theorem 2),

(b) $\quad L^{*\alpha\beta}_{\ ij} \overset{\text{def}}{=} L^{\alpha\beta}_{ij} - \frac{1}{m} g^{\alpha\beta}\, g_{\gamma\delta}\, L^{\gamma\delta}_{ij} = 0,$

(c) $\quad L^{\alpha\beta}_{ij} = L^{\beta\alpha}_{ij}, \quad L^{\alpha\beta}_{ij} L^{\gamma\delta}_{hk} = L^{\gamma\delta}_{ij} L^{\alpha\beta}_{hk},$

$$L^{[\alpha_1\ [\beta_1}_{i_1\ \ j_1} L^{\alpha_2\ \beta_2}_{i_2\ j_2} \cdots L^{\alpha_m]\ \beta_m]}_{i_m\ \ j_m} \neq 0.$$

There is a non-trivial remarkable subclass of the submetric class. An areal space of this class, called the *semi-metric class*, is characterized by the relation

$$L^{(\alpha\beta)}_{ij} = \overset{\prime\prime}{g}_{ij}\, g^{\alpha\beta}, \qquad 2 \leqq m \leqq n-2.$$

Then we have the following interesting theorem under the assumption that the space under consideration is regular, *i.e.*, det.$|g_{ij}| \neq 0$.

Theorem 6. *In a space of the semi-metric class, there exists a well-determined scalar field* $\wedge(x, p)\neq 0$, *which is a rational function of the fundamental function* F *and its successive derivatives of order not higher than* 4 *with respect to* p^i_α, *such that the new fundamental function* $\bar{F}=\wedge^{-1}F$ *determines a Riemannian metric on the underlying manifold, i.e., the associated areal space with the fundamental function* $\bar{F}$ *is a Riemannian space.*

Now let us consider the *characteristic m-tensor*

$$J_{i[m],j[m]}\overset{\text{def}}{=}g_{i[m],j[m]}-m!g_{[i_1[j_1}\, g_{i_2 j_2}\cdots\cdots g_{i_m]j_m]},$$

which is different from zero for the spaces of the non-metric class, and satisfies the relation

$$J_{i[m],j[m]}\, p^{j[m]}=0,$$

when g_{ij} is the normalized metric tensor. As the generalization of the usual partial derivatives, we define the partial derivatives of a function $\Phi(p^{i[m]})$ with respect to $p^{i[m]}$ as follows :

$$\Phi_{;i[m]}=m\Phi^{[1}_{[i_1}\, p^{2}_{i_2}\,\cdots\, p^{m]}_{i_m]},\qquad \Phi^{\alpha}_{i}=\frac{\partial\Phi}{\partial p^{i}_{\alpha}},$$

where $\Phi(p^{i[m]})$ is any homogeneous function of degree ρ in the $p^{i[m]}$'s.

Then the following relations hold :

$$\Phi_{;i[m]}\, p^{i[m]}=\Phi^{\alpha}_{i}\, p^{i}_{\alpha}=m\rho\Phi,$$

$$\Phi_{;i[m]}\, dp^{i[m]}=d\Phi,$$

$$p^{i[m]}{}_{;j[m]}=\delta^{[i_1}_{[j_1}\, p^{i_2}_{j_2}\cdots p^{i_m]}_{j_m]},$$

On account of these relations, we have, putting $l^{h[m]}=\dfrac{1}{F}p^{h[m]}$,

$$J_{o,j[m],k[m]}\equiv\frac{1}{m!}l^{h[m]}\, J_{h[m],j[m];k[m]}$$

$$=-\frac{1}{(m-1)!\,F}\, J_{h[m],j[m]}\,\delta^{[h_1}_{[k_1}\, p^{h_2}_{k_2}\,\cdots\, p^{h_m]}_{k_m]},$$

from which it follows that

$$J_{o,j[m],k[m]}\, l^{j[m]}=0,$$

$$J_{o,j[m],k[m]}=-m^2(m!)\, Fp^{[1}_{[j_1}\cdots p^{m-1}_{j_{m-1}}\, L^{*}{}^{m][m}_{j_m][k_m}\, p^{1}_{k_1}\cdots p^{m-1]}_{k_{m-1}]},$$

Hence

$$J_{o,}{}^{i[m]}{}_{k[m]}\equiv g^{i_1j_1}\, g^{i_2j_2}\cdots g^{i_mj_m}\, J_{o,j[m],k[m]}$$

$$=-m^2(m!)F^{-1}\, p^{[i_1}_{[1}\cdots p^{i_{m-1}}_{m-1}\, g_{m]\alpha}\, g^{i_m]j}\, L^{*\alpha[m}_{j\;[k_m}\, p^{1}_{k_1}\cdots p^{m-1]}_{k_{m-1}]}$$

$$=m\, l^{j[i_1\dots i_{m-1}}\, J^{i_m]}{}_{j,k[m]},$$

where

$$J^{i}{}_{h,k[m]} \equiv -m\, p^{\beta}_{h}\, g_{\beta\alpha}\, g^{ij} L^{*\alpha\,[m}_{\ j\,[k_m}\, p^{1}_{k_1} \dots p^{m-1]}_{k_{m-1}]}$$

$$= -m\, g_{hl}\, g^{ij}\, p^{l}_{\alpha}\, L^{*\alpha[m}_{\ j[k_m}\, p^{1}_{k_1} \dots p^{m-1]}_{k_{m-1}]},$$

whence evidently

$$J^{h}{}_{h,k[m]} = 0.$$

We shall introduce the absolute differentiation of a contravariant vector X^i in the form

$$DX^{i} = dX^{i} + \left(\Gamma^{i}_{jh}\, dx^{h} + C^{i}_{j,k[m]}\, dp^{k[m]}\right) X^{j},$$

where the functions Γ^{i}_{jh} and $C^{i}_{j,k[m]}$ should be subjected to the condition

$$Dg_{ij} = 0,$$

that is

$$\Gamma_{ij,h} + \Gamma_{ji,h} = \frac{\partial g_{ij}}{\partial x^{h}},$$

$$C_{ij,k[m]} + C_{ji,k[m]} = g_{ij;k[m]},$$

where

$$\Gamma_{ij,h} \equiv g_{ik}\, \Gamma^{k}_{jh}, \quad C_{ij,k[m]} \equiv g_{ih}\, C^{h}_{j,k[m]}.$$

Hence we may put

$$C_{ij,k[m]} \equiv \tfrac{1}{2}\, g_{ij;k[m]} + \frac{1}{F}\pi_{ij,k[m]}, \tag{24}$$

$\pi_{ij,k[m]}$ being the skew-symmetric part of $C_{ij,k[m]}$ with respect to the indices i and j.

Let us put

$$\Gamma^{i[m]}_{o\ \ h} \equiv m\, l^{k[i_2 \dots i_m}\, \Gamma^{i_1]}_{kh}, \quad C^{i[m]}_{o\ \ ,k[m]} \equiv m\, l^{h[i_2 \dots i_m}\, C^{i_1]}_{h,k[m]},$$

$$l^{i[m]} = \frac{1}{F} p^{i[m]}, \quad \pi^{i[m]}_{o\ \ ,k[m]} \equiv m\, l^{h[i_2 \dots i_m}\, g^{i_1]l}\, \pi_{lh,k[m]};$$

then we have the absolute differential of the unit m-vector $l^{i[m]}$:

$$Dl^{i[m]} = dl^{i[m]} + \Gamma^{i[m]}_{o\ \ h}\, dx^{h} + C^{i[m]}_{o\ \ ,k[m]}\, dp^{k[m]}$$

and from (24) it follows

$$C_{o}{}^{i[m]}{}_{,k[m]} = m\, l^{h[i_2 \dots i_m}\, g^{i_1]l} \left(\tfrac{1}{2}\, g_{lh;k[m]} + \frac{1}{F}\pi_{lh,k[m]}\right)$$

$$=\tfrac{1}{2}\,g^{i_1j_1}g^{i_2j_2}\cdots g^{i_mj_m}l^{h[m]}\Big(g_{[j_1\,[h_1}g_{j_2h_2}\cdots g_{j_m]h_m]}\Big)_{;k[m]}+\frac{1}{F}\,\pi_o{}^{i[m]}{}_{,k[m]}$$

$$=-\tfrac{1}{2}\,J_o{}^{i[m]}{}_{,k[m]}+\frac{1}{F}\,\pi_o{}^{i[m]}{}_{,k[m]}\,.$$

Therefore we can choose the function $\pi_{jh,k[m]}$ in such a way that

$$-\frac{1}{F}\;\pi_{jh,k[m]}=g_{j[i}J^{i}_{h],k[m]}\;;$$

then it is easily seen that

$$C_o{}^{i[m]}{}_{,k[m]}=0\,,$$

because

$$-\frac{1}{F}\,p^{j}_{\alpha}\,\pi_{jh,k[m]}=\tfrac{1}{2}\,p^{j}_{\alpha}\left(g_{ji}J^{i}_{h,k[m]}-g_{hi}J^{i}_{j,k[m]}\right)$$

$$=-\tfrac{1}{2}p^{j}_{\alpha}\,g_{hi}J^{i}_{j,k[m]}\,.$$

Accordingly we obtain

$$D\,l^{i[m]}=dl^{i[m]}+\Gamma^{\;i\,[m]}_{o\;\;\;h}\,dx^{h},$$

which gives the base connection in our space, and we can write the absolute differential of a contravariant vector :

$$DX^{i}=dX^{i}+\left(\Gamma^{*i}_{jh}dx^{h}+\frac{1}{F}\,C^{i}_{j,k[m]}Dp^{k[m]}\right)X^{j},$$

because $C^{i}_{j,k[m]}\,p^{k[m]}=0$ and

$$Dp^{k[m]}=FDl^{k[m]}-l^{k[m]}\,dF\,,$$

where
$$\Gamma^{*i}_{jh}\equiv\Gamma^{i}_{jh}-C^{i\alpha}_{j,k}\,p^{l}_{\alpha}\,\Gamma^{k}_{lh}\,.$$

The unknown functions Γ^{i}_{jh} can be determined from the system of linear equations :

$$\Gamma^{*i}_{jh}=\Gamma^{*i}_{hj}\,,$$

$$g_{ik}\Gamma^{k}_{jh}+g_{jk}\Gamma^{k}_{ih}=\frac{\partial g_{ik}}{\partial x^{h}}\,,$$

if the determinant of this system of linear equations is not equal to zero.

Thus the absolute differential can be determined uniquely; however, it can be noticed that $Dg_{ij}=0$ but $Dg_{i[m],j[m]}$ is not always equal to zero.

4. We are now in a position to proceed into the theory of areal spaces of the general type.

Let g_{ij} be a non-singular symmetric covariant tensor of second order satisfying the following :

$$(25)\qquad \text{det.}\left|g_{\alpha\beta}\right| = F^2 \quad \left(g_{\alpha\beta} = g_{ij}p^i_\alpha\, p^j_\beta\right),$$

$$(26)\qquad g_{ij}\, q^i_h\, p^j_\alpha = 0,\quad q^i_h = \delta^i_h - p^i_h,\ p^i_h = p^i_\alpha p^\alpha_h,$$

$$(27)\qquad g_{ij}\Big|^{\gamma}_{k}\, q^i_h\, p^j_\gamma = 0,$$

$$(28)\qquad g^{ij}\Big|^{\gamma}_{k}\, q^k_l\, p^\delta_j = 0,$$

nevertheless we cannot assume its unique existence. As (27) and (28) can be rewritten in the forms respectively :

$$(29)\qquad mg''_{ij} = L^{\alpha\beta}_{ij}\, g_{\alpha\beta},$$

$$(30)\qquad (n-m)\, g^{\alpha\beta} = L^{\alpha\beta}_{ij}\, g^{ij},$$

we have a system of algebraic relations to define g_{ij}. Let us consider a pair of the $(n-m)$-ples q^i_λ and q^λ_i such that

$$q^i_\mu\, q^\lambda_i = \delta^\lambda_\mu,\ q^i_\mu\, p^\alpha_i = 0,\qquad q^\lambda_i\, p^i_\alpha = 0;$$

then we see

$$q^i_\lambda\, q^\lambda_j = q^i_j = \delta^i_j - p^i_j,\quad g_{ij}q^i_\lambda q^j_\mu \overset{\text{def}}{=} g_{\lambda\mu},\quad g^{ij}q^\lambda_i q^\mu_j \overset{\text{def}}{=} g^{\lambda\mu}$$

and the metric tensor g_{ij} is completely characterized by

$$g_{ij} = g_{\alpha\beta}p^\alpha_i p^\beta_j + g_{\lambda\mu}q^\lambda_i q^\mu_j$$

on account of (25) and

$$(31)\qquad m\, g_{\lambda\mu} = L^{\alpha\beta}_{\lambda\mu} g_{\alpha\beta},$$

$$(32)\qquad (n-m)\, g^{\alpha\beta} = L^{\alpha\beta}_{\lambda\mu}\, g^{\lambda\mu},$$

where $L^{\alpha\beta}_{\lambda\mu} = L^{\alpha\beta}_{ij} q^i_\lambda q^j_\mu$, if we assume det. $|g_{ij}| \neq 0$. Since (31) and (32) are homogeneous with respect to $g_{\alpha\beta}$ and $g_{\lambda\mu}$, they admit at least one non-trivial solution, because (31) and (32) give rise to the same relation $L^{\alpha\beta}_{\lambda\mu} g_{\alpha\beta} g^{\lambda\mu} = m(n-m)$; hence we can assume without loss of generality that their resultant vanishes identically. Hence for the solutions $\rho g_{\alpha\beta}$ and $\rho g_{\lambda\mu}$ of (31) and (32) under the assumption that det. $|\rho g_{\alpha\beta}|$ det. $|\rho g_{\lambda\mu}| \neq 0$, we have a metric tensor g_{ij}, whose components are independent of the choice of q^i_λ and q^λ_i. Thus we have the complete set of metric tensors $g^{(1)}_{ij}, g^{(2)}_{ij}, \ldots, g^{(M)}_{ij}$ whose components are algebraic functions of F, p^α_i and $L^{\alpha\beta}_{ij}$. However these tensors may not be real-valued in general; in fact, if g_{ij} is a metric tensor, then $\epsilon^k g_{ij}$, $\epsilon^k \bar{g}_{ij}$ $(k=0, 1, 2, \ldots, m-1)$ are also metric tensors, ϵ being a primitive m-th root of unity and $\bar{g}_{ij}$ the complex conjugate tensor of g_{ij}. The set of all the metric tensors is divided into mutually disjoint subsets, each one of which is consisting of $\epsilon^k g_{ij}$ and $\epsilon^k \bar{g}_{ij}$ $(k=0, 1, \ldots, m-1)$, g_{ij} being a representative of the subset. Such a subset is called a *class of metric tensors* and denoted by $\{g_{ij}\}$. Put

$$G_{ij,kh}=\tfrac{1}{2}(g_{ij}\bar{g}_{kh}+\bar{g}_{ij}g_{kh}),\quad H_{ij,kh}=\frac{1}{2\sqrt{-1}}(g_{ij}\bar{g}_{kh}-\bar{g}_{ij}g_{kh}).$$

Then $G_{ij,kh}$ and $H_{ij,kh}$ are real quantities and invariant under a change $g_{ij}\to\epsilon^k g_{ij}$. On the other hand, by a change $g_{ij}\to\bar{g}_{ij}$, $G_{ij,kh}$ and $H_{ij,kh}$ turn into $G_{ij,kh}$ and $-H_{ij,kh}$. Thus $G_{ij,kh}$ depends only on the class of metric tensors $\{g_{ij}\}$.

On the other hand, the present author succeeded in determining the connection uniquely and in finding the concrete forms of its connection parameters in the areal space of the most general type in the simplest forms made of the fundamental function F and its derivatives.

Put

$$g_{ij}{}^{hk} \overset{\text{def}}{=} g_{i\,i_2\cdots i_m,\,j\,j_2\cdots j_m}\, g^{h\,i_2\cdots i_m,\,k\,j_2\cdots j_m},$$

$$\Phi_{ij}{}^{hk} \overset{\text{def}}{=} \binom{n-2}{m-1}^{-1}\left\{((m-1)!)^{-2} g_{ij}{}^{hk}-\binom{m-2}{m-1}\delta_i^h\delta_j^k\right\},$$

$$\Lambda_{ij}{}^{hk} \overset{\text{def}}{=} \Phi_{(ij)}{}^{hk},$$

$$\Lambda_{ij}{}^{\alpha\beta} \overset{\text{def}}{=} \Lambda_{ij}{}^{hk} p_h^\alpha p_k^\beta.$$

Under the assumption that the mn-rowed det. $|\Lambda_{ij}{}^{\alpha\beta}|\neq 0$, we introduce $\Lambda^{hj}{}_{\gamma\beta}$ by

$$\Lambda_{ij}{}^{\alpha\beta}\Lambda^{hj}{}_{\gamma\beta}=\delta_i^h\delta_\gamma^\alpha$$

and put $\overset{*}{\Lambda}_{ij}{}^{\alpha\beta}=F^{\frac{2}{m}}\Lambda_{ij}{}^{\alpha\beta}$, $\Lambda^{*ij}{}_{\alpha\beta}=F^{-\frac{2}{m}}\Lambda^{ij}{}_{\alpha\beta}$.

Then we have the connection

$$\delta V^i=dV^i+\Gamma^{*i}_{jk}V^j dx^k+C^{i}_{j,}{}^{\delta}_{h}V^j\,\delta p^h.$$

where

$$\delta p^i_\alpha=q^i_j(dp^j_\alpha+B^j_{\alpha k}dx^k),$$

$$\Gamma^{*i}_{jh}=\gamma^i_{jh}-C^{\times i\gamma}_{j,\,h}B^k_{\gamma h}-C^{\times i\gamma}_{h,\,k}B^k_{\nu j}+C^{\times ir\gamma}_{jh,\,k}B^k_{\gamma r},$$

$$C^{i\gamma}_{jk}=C^{\times i\gamma}_{j,\,k}-p^h_j\,C^{\times i\gamma}_{h,\,k},$$

$$C^{\times hr\gamma}_{ij,\,k}=\frac{1}{2m}\Lambda^{*hr}{}_{\alpha\beta}\overset{*}{\Lambda}_{ij}{}^{\alpha\beta}\big|^{\gamma}_{k}.$$

We shall introduce an affine connection based on a not necessarily real valued metric tensor g_{ij}, such that an absolute differential of any vector is real and the connection will be determined depending only on $\{g_{ij}\}$.

Let an absolute differential of a contravariant vector V^i be written in the form

$$\delta V^i = dV^i + \overset{*}{\Gamma}{}^i_{jk} V^j dx^k + C^{i\ \delta}_{j,\ h} V^j \delta p^h_\delta,$$

where the base connection is given by

$$\delta p^i_\alpha = q^i_j\, dp^j_\alpha + B^i_{\alpha k}\, d x^k,$$

satisfying the conditions :

$$\delta g_{ij} = 0, \qquad \overset{*}{\Gamma}{}^i_{jk} = \overset{*}{\Gamma}{}^i_{kj},$$

$$B^i_{\alpha k} = \overset{*}{\Gamma}{}^h_{jk}\, q^i_h\, p^j_\alpha, \qquad C^{i\ \delta}_{j,\ h}\, q^s_i\, p^j_\alpha = 0,$$

$$C^{i\ \delta}_{j,h}\Big|^{[\gamma}_{k}\, g^{\alpha]\beta}\, q^k_l\, p^j_\beta = 0, \qquad C^{i\ \delta}_{j\ h}\Big|^{\gamma}_{[k}\, g''_{s]l}\, p^j_\gamma = 0.$$

Then the connection parameters are expressed as

$$\overset{*}{\Gamma}{}^i_{jk} = \left\{ {i \atop jk} \right\} - \overset{\times}{C}{}^{i\ \delta}_{j,h}\, B^h_{\delta k} - \overset{\times}{C}{}^{i\ \delta}_{k,h}\, B^h_{\delta j} + \overset{\times}{C}{}^{ir\ \delta}_{jk,h}\, B^h_{\delta r},$$

$$\overset{\times}{C}{}^{ir\ \delta}_{jk,h} = \tfrac{1}{2} g^{ir}\, g_{jk}\Big|^{\delta}_{h}, \qquad \overset{\times}{C}{}^{i\ \delta}_{j,h} = \overset{\times}{C}{}^{ir\ \delta}_{jk,h},$$

$$C^{i\ \delta}_{j,h} = \overset{\times}{C}{}^{i\ \delta}_{j,h} + g^{il}\, \pi_{lj,h}{}^{\delta}, \qquad \pi_{ij,h}{}^{\delta} = -p^r_\beta\, g_{r[i}\, \overset{*}{L}{}^{\beta\delta}_{j]h}.$$

Let us denote the other absolute differential $\bar{\delta} V^i$ obtained by the same way in terms of the metric tensor $\bar{g}_{ij}$; then we can easily see that

$$\bar{\delta} V^i = \bar{\delta} \bar{V}^i.$$

On account of this fact, the third connection is defined by

$$DV^i = \tfrac{1}{2}\left(\delta V^i + \bar{\delta} V^i\right) = dV^i + \tilde{\overset{*}{\Gamma}}{}^i_{jk} V^j dx^k + \tilde{C}^{i\ \delta}_{j,h} V^j Dp^h_\delta,$$

where

$$Dp^i_\alpha = \tfrac{1}{2}\left(\delta p^i_\alpha + \bar{\delta} p^i_\alpha\right) = q^i_j\, dp^j_\alpha + \tilde{B}^i_{\alpha k}\, dx^k,$$

which is called the *canonical connection in terms of* $\{g_{ij}\}$.

The torsion and curvature tensors are obtained from

$$(D_2 D_1 - D_1 D_2)\, x^i = S^i_{jk}\left[d_1 x^j,\ d_2 x^k\right] + \tilde{C}^{i\ \delta}_{j,k}\left[d_1 x^j,\ D_2 p^k_\delta\right],$$

$$(D_2 D_1 - D_1 D_2)\, V^i = \Big\{\tfrac{1}{2} R^i_{jkh}\left[d_1 x^k,\ d_2 x^h\right] + M^{i\ \ \gamma}_{jk,h}\left[d_1 x^k,\ D_2 p^h_\gamma\right]$$

$$+\, C^{i\ \ \gamma\ \delta}_{j,\ h,k}\left[D_1 p^h_\gamma,\ D_2 p^k_\delta\right]\Big\} V^j$$

and have the forms :

$$S^i_{jk} = \tfrac{1}{2}\left(\tilde{\overset{*}{\Gamma}}{}^i_{jk} - \tilde{\overset{*}{\Gamma}}{}^i_{kj}\right),$$

$$R^{i}{}_{j,kr}=\left(\delta^{i}_{h}\,\delta^{s}_{j}-\tilde{C}^{i\ \delta}_{j,\,h}\,p^{s}_{\delta}\right)\left(\frac{\partial\tilde{\Gamma}^{*h}_{sk}}{\partial x^{r}}-\frac{\partial\tilde{\Gamma}^{*h}_{sr}}{\partial x^{k}}-\tilde{\Gamma}^{*h}_{sk}\Big|^{\epsilon}_{t}\,\tilde{B}^{t}_{\epsilon\,r}+\tilde{\Gamma}^{*h}_{sr}\Big|^{\epsilon}_{t}\,\tilde{B}^{t}_{\epsilon k}\right.$$
$$\left.+\tilde{\Gamma}^{*t}_{sk}\,\tilde{\Gamma}^{*h}_{tr}-\tilde{\Gamma}^{*t}_{sr}\,\tilde{\Gamma}^{*h}_{tk}\right),$$

$$M^{i\ \ \gamma}_{jk,\,h}=\tilde{\Gamma}^{*i}_{jk}\Big|^{\gamma}_{h}-\frac{\partial}{\partial x^{k}}\tilde{C}^{i\,\gamma}_{j,h}+\tilde{C}^{i\,\gamma}_{j,h}\Big|^{\delta}_{r}\,\tilde{B}^{r}_{\delta k}+\tilde{C}^{i\,\gamma}_{s,h}\,\tilde{\Gamma}^{*s}_{jk}$$
$$-\tilde{C}^{s\,\gamma}_{j,h}\,\tilde{\Gamma}^{*i}_{sk}+\tilde{C}^{i\,\delta}_{j,r}\,\tilde{B}^{\ r}_{\delta k}\Big|^{\gamma}_{h},$$

$$C^{i\,\gamma\,\delta}_{j,h,r}=\tilde{C}^{i\,\gamma}_{j,h}\Big|^{\delta}_{r}-\tilde{C}^{i\,\delta}_{j,r}\Big|^{\gamma}_{h}+\tilde{C}^{s\,\gamma}_{j,h}\,\tilde{C}^{i\,\delta}_{s,r}-\tilde{C}^{s\,\delta}_{j,\,r}\,\tilde{C}^{i\,\gamma}_{s,h},$$

$$C^{\times i\gamma}_{\ \ j,k}=C^{\times\,hi\,\gamma}_{\ \ \ hj,k}\,,$$

$$\gamma^{h}_{ij}=\frac{1}{2m}\Lambda^{*hr}{}_{\alpha\beta}\left\{\frac{\partial\Lambda^{*}{}_{rj}{}^{\alpha\beta}}{\partial x^{i}}+\frac{\partial\Lambda^{*}{}_{ir}{}^{\alpha\beta}}{\partial x^{j}}-\frac{\partial\Lambda^{*}{}_{ij}{}^{\alpha\beta}}{\partial x^{r}}\right\},$$

$$B^{p}_{\sigma q}=\gamma^{i}_{jk}\,p^{j}_{\alpha}\,\tilde{W}^{pk\alpha}_{iq\sigma}\,,\qquad\tilde{W}^{pk\alpha}_{iq\delta}\,W^{ir\gamma}_{sk\alpha}=\delta^{p}_{s}\;\delta^{r}_{q}\;\delta^{\gamma}_{\delta},$$

$$W^{ir\gamma}_{hk\alpha}=\left(C^{\times i\,\gamma}_{\ \ k,h}\,\delta^{r}_{j}+C^{\times i\ \gamma}_{\ \ j,h}\,\delta^{r}_{k}-C^{\times ir\,\gamma}_{\ \ jk,h}\right)p^{j}_{\alpha}+\delta^{r}_{\alpha}\;\delta^{i}_{h}\;\delta^{r}_{k}\,.$$

We state the following theorem :

Theorem 7. *If the space is of the submetric class and the relations*

$$\Lambda_{ij}{}^{hk}=g_{ij}\,g^{hk}\ \textit{and}\ F^{2}=\det.\left|g_{\alpha\beta}\right|$$

hold good, then

$$\Lambda^{*hj}{}_{\alpha\beta}\,\frac{\partial\Lambda^{*}{}_{ij}{}^{\alpha\beta}}{\partial x^{k}}=g^{hj}\frac{\partial g_{ij}}{\partial x^{k}},$$

$$C^{hr\,\gamma}_{ij,\,k}=g^{hr}\,\frac{\partial g_{ij}}{\partial p^{k}_{\gamma}},$$

and γ^{h}_{ij} *is the Christoffel symbol in usual sense.*

Hence the introduced connection parameters coincide with those stated before in the case of the submetric class.

Remark. There are many other topics, for example, theory of subspaces, variational problems and so on. Some were solved in this field but there remain very many unsolved problems which are looking forward further investigations of young talented research workers.

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ON HYPERCONNECTIONS

By

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The hyperconnection in the sense of the present paper was, in our knowledge, first considered by R. König [4]. The hyperconnection in 5-dimensional vector bundle over a 4-dimensional base space was used by A. Einstein and W. Mayer [2] to establish the so-called unified field theory of gravitation and electromagnetism. B. Hoffmann [3] considered the hyperconnection in 6-dimensional vector bundle over a 4-dimensional base space to get more general unified field theory. The theory of hyperconnection was further generalized in this direction by A. D. Michal and J. L. Botsford [5] and one of the present authors [8].

More recent contribution to the theory of hyperconnection was given by S. Bochner [1] who discussed the subject by his method famous in "Curvature and Betti numbers."

In the present paper, we develop the theory of hyperconnection from a little bit modern point of view. We hope this is the first of a series of our papers in which we shall discuss integral formulas and their application, groups of transformations, the representation of projective, conformal or more general connection.

1. Hypertangent bundles.

Let M be an n-dimensional differentiable manifold. If we denote by $T(M)$ the tangent bundle of M, $T(M)$ is an n-dimensional differentiable vector bundle over M. Its structure group is the linear group $L(n)$, *i.e.*, the group of all linear transformations in the fibre space V_n, which is a vector space of n dimensions.

Let $\mathfrak{T}(M)$ be an N-dimensional differentiable vector bundle over M. Then $\mathfrak{T}(M)$ is a differentiable fibre bundle over M, which has a vector space V_N of N dimensions as its fibre space. The structure group of $\mathfrak{T}(M)$ is the linear group $L(N)$, *i.e.*, the group of all linear transformations in the N-dimensional vector space V_N. If we suppose that the structure group of $\mathfrak{T}(M)$ is differentiablly reducible to a subgroup of $L(N)$ which leaves invariant an n-dimensional subspace of V_N, then there exists in $\mathfrak{T}(M)$ a subbundle $T'(M)$ which is an n-dimensional vector bundle [7]. Assuming the subbundle $T'(M)$ to be differentiablly equivalent to the tangent bundle $T(M)$ of M [7], we call the vector bundle $\mathfrak{T}(M)$ an N-dimensional *hypertangent bundle* over the *base space* M. The subbundle $T'(M)$ being equivalent to the tangent bundle $T(M)$, if $T(M)$ is identified with $T'(M)$, the tangent bundle can be considered as a subbundle which is differentiablly imbedded in $\mathfrak{T}(M)$. Although there might be many ways to imbed $T(M)$ in $\mathfrak{T}(M)$, we suppose that there is given a fixed imbedding of $T(M)$ in $\mathfrak{T}(M)$. Then, if we denote by T_P the tangent space of M at a point P belonging to M and $\mathfrak{T}_P$ the fibre of $\mathfrak{T}(M)$ at P, T_P is a subspace of $\mathfrak{T}_P$. We call the fibre $\mathfrak{T}_P$ the *hypertangent space* at P.

Since the tangent bundle $T(M)$ is imbedded in $\mathcal{J}(M)$ as a subbundle, there exists a subbundle $\bar{T}(M)$, which is complementary to $T(M)$. That is to say, the subbundle $\bar{T}(M)$, is an m-dimensional vector bundle over M, m being equal to $N-n$, and the hypertangent space $\mathcal{J}_P$ is the direct product $T_P \times \bar{T}_P$, where $\bar{T}_P$ is the fibre of $\bar{T}(M)$ at the point P. The subbundle $\bar{T}(M)$ is called a *vertical vector bundle*. The tangent space T_P and the fibre $\bar{T}_P$ of $\bar{T}(M)$ are respectively called the *tangent plane* and the *vertical plane* in the hypertangent space $\mathcal{J}_P$ at each point P of M. Corresponding to the choice of the vertical vector bundle, we can construct in $\mathcal{J}(M)$ fields of vertical planes in several ways.

2. **Tensor fields.**

Let $\mathcal{J}(M)$ be a hypertangent bundle over M. Any element of $\mathcal{J}(M)$ is called a (contravariant) *vector*. A cross-section and a local cross-section of $\mathcal{J}(M)$ are called a (contravariant) *vector field* and a *local vector field* respectively. We denote by $\mathcal{J}^*(M)$ the vector bundle dual to the hypertangent bundle $\mathcal{J}(M)$ and call $\mathcal{J}^*(M)$ the *hypercotangent bundle* associated with $\mathcal{J}(M)$. Any element of $\mathcal{J}^*(M)$ is called a *covector* (*covariant vector*). A cross-section and a local cross-section of $\mathcal{J}^*(M)$ are called a *covector field* and a *local covector field*, respectively.

In a cubic coordinate neighborhood U of M, we consider N local vector fields $\mathbf{e}_1$, $\mathbf{e}_2, \ldots, \mathbf{e}_N$ linearly independent at each point of U. The ordered set $\{\mathbf{e}_h\}=\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ of these local vector fields is called a family of N-frames, or briefly an *N-frame*, in U. Any vector field X, being a linear combination of $\mathbf{e}_h$, is expressed in U as*)

$$X=X^h \mathbf{e}_h,$$

where $X^h=X^h(x^1, x^2, \ldots, x^N)$ are N functions of coordinates x's defined in U and the functions X^h are called the *components* of the vector field X with respect to the N-frame $\{\mathbf{e}_h\}$.

An N-frame $\{\mathbf{e}_h\}$ being given in U, there exist in U local covector fields $\mathbf{f}^1$, $\mathbf{f}^2, \ldots \mathbf{f}^N$ such that

$$\mathbf{f}^i(\mathbf{e}_h)=\delta^i_h.$$

The ordered set $\{\mathbf{f}^i\}=\{\mathbf{f}^1, \mathbf{f}^2, \ldots, \mathbf{f}^N\}$ of these local covector fields is called a family of N-coframe, or briefly an *N-coframe*, which is dual to the frame $\{\mathbf{e}_h\}$. Any covector field η, being a linear combination of $\mathbf{f}^i$, is expressed in U as

$$\eta=\eta_i \mathbf{f}^i,$$

where $\eta_i=\eta_i(x^h)$ are N functions of coordinates x's defined in U and the function η_i are called the *components* of the covector field η with respect to the given frame $\{\mathbf{e}_h\}$.

Let X and η be respectively a vector field with components X^h and a covector field with components η_i with respect to a frame $\{\mathbf{e}_h,\}$ the inner product $\eta(X)$ of X and η is a scalar function in M, which is given in U by

$$\eta(X)=X^h \eta_h.$$

---

* The indices h, i, j, k, run over the range $\{1, 2, \ldots, N\}$, and the indices a, b, c, d, e the range $\{1, 2, \ldots, n.\}$

We shall now introduce the *hypertensor bundle* associated with the hypertangent bundle $\mathcal{T}(M)$. For example, the hypertensor bundle $\mathcal{T}^1_2(M)$ of (1, 2)-type is by definition the tensor product $\mathcal{T}^*(M)\times\mathcal{T}^*(M)\times\mathcal{T}(M)$. Any element of the bundle $\mathcal{T}^1_2(M)$ is called a *tensor* of (1, 2)-type. A cross-section and a local cross-section of the bundle $\mathcal{T}^1_2(M)$ are called respectively a *tensor field* and a *local tensor field* of (1, 2)-type. In a similar way, we can define the hypertensor bundle $\mathcal{T}^p_q(M)$ of (p, q)-type and tensor fields of (p, q)-type.

Let T be a tensor field, say, of (1, 2)-type. Then, T is written in U

$$T=T_{ji}{}^h\, \mathbf{f}^j \times \mathbf{f}^i \times \mathbf{e}_h$$

as a linear combination of $\mathbf{f}^j \times \mathbf{f}^i \times \mathbf{e}_h$, where the functions $T_{ji}{}^h$ of coordinates x'*s* are called the *components* of T with respect to the N-frame $\{\mathbf{e}_h\}$.

Take a tensor field T of (1, 2)-type, two vector fields X, Y and a covector field η. The scalar function $T(X, Y, \eta)$ is by definition given by

$$T(X,Y,\eta)=T_{ji}{}^h X^j Y^i \eta_h$$

in U, where $T_{ji}{}^h$, X^h, Y^h, η_i are the components of T, X, Y, η respectively. The sets of functions $T_{ji}{}^h X^j Y^i$, $T_{ji}{}^h X^j \eta_h$, $T_{ji}{}^h X^j$ and $T_{ji}{}^h \eta_h$ are respectively in U the components of a vector field, a covector field, a tensor field of (1, 1)-type and a tensor field of (0, 2) type with respect to the N-frame $\{\mathbf{e}_h\}$.

3. Hyperconnection.

Let $\mathcal{T}(M)$ be an N-dimensional hypertangent bundle over the base space M. Since the structure group of $\mathcal{T}(M)$ is the linear group $L(N)$, there exists an infinitesimal connection ω in the principal bundle $B(M, L(N))$ associated with $\mathcal{T}(M)$ [6]. The connection ω is called a *hyperconnection* in the hypertangent bundle $\mathcal{T}(M)$.

We take now an N-frame $\{\mathbf{e}_h\}$ in a coordinate neighborhood U of M. Then, since $\{\mathbf{e}_h\}$ is a local cross-section of B over U, the hyperconnection ω is analytically expressed by a matrix

$$(\ \omega_i^h\ )$$

of differential forms ω_i^h defined in U. The geometrical variation of the N-frame $\{\mathbf{e}_h\}$ with respect to the hyperconnection ω is given by

$$\mathrm{D}\mathbf{e}_i=\omega_i^h \mathbf{e}_h \tag{3.1}$$

If we consider an N-frame $\{\mathbf{e}_{h'}\}$ in a coordinate neighborhood U' of M, then we have in $U \cap U'$ two N-frames $\{\mathbf{e}_h\}$ and $\{\mathbf{e}_{h'}\}$, which are related to each other by*)

$$\mathbf{e}_{h'}= A^h_{h'}\, \mathbf{e}_h, \tag{3.2}$$

where the matrix $(A^h_{h'})$ is a function in $U \cap U'$, and non-singular. If the hyperconnection ω is represented by matrices (ω_i^h) and $(\omega_{i'}^{h'})$ with respect to N-frames $\{\mathbf{e}_h\}$ and $\{\mathbf{e}_{h'}\}$ respectively, we have the following transformation law

$$\omega_{i'}^{h'} = A^i_{i'} A^{h'}_h \omega_i^h + (dA^i_{i'})\, A^{h'}_i \tag{3-3}$$

---

* The indices h', i', j', k', l' run over the range $\{1,' 2,' \dots, N'\}$.

in $U \cap U'$, by virtue of the right invariance of ω, where the matrix $(A_h^{h'})$ is the inverse of the matrix $(A_{h'}^h)$

Putting

$$\omega_i^h = \Gamma_{c\,i}^{\,h}\, dx^c, \tag{3.4}$$

we call the functions $\Gamma_c{}^h{}_i$ the *components* of the hyperconnection ω with respect to the N-frame $\{e_h\}$ in U. If we denote by $\Gamma_c{}^{h'}{}_{i'}$ the components of ω with respect to another N-frame $\{e_{h'}\}$ we have the law of transformation

$$\Gamma_c{}^{h'}{}_{i'} = \Gamma_c{}^h{}_l A_{i'}^l A_h^{h'} + (\partial_c A_{i'}^l) A_l^{h'} \tag{3.5}$$

as a direct consequence of (3.3) and (3.4), where ∂_c denotes $\partial/\partial x^c$.

For a global vector field $X = X^h e_h$, we define the covariant derivative of X^h by

$$\nabla_c X^h = \partial_c X^h + \Gamma_c{}^h{}_i X^i, \tag{3.6}$$

which determines a global vector field in $\mathcal{T}(M)$ with respect to the index h and a global covector field in the base space M with respect to the index c.

We take a global tensor field $T = T_i{}^h e_h \times f^i$ of (1, 1)-type, for example. We define the covariant derivative of $T_i{}^h$ by

$$\nabla_c T_i{}^h = \partial_c T_i{}^h + \Gamma_c{}^h{}_k T_i{}^k - \Gamma_c{}^k{}_i T_k{}^h, \tag{3.7}$$

which determines a global tensor field of (1, 1)-type in $\mathcal{T}(M)$ with respect to the indices h, i and a global covector field in the base space M with respect to the index c. We can define the covariant derivative of tensor field of any type, in a quite similar way.

4. Induced connection.

We suppose now that there is given in the hypertangent bundle $\mathcal{T}(M)$ a vertical vector bundle, *i.e.*, a field of vertical planes $\bar{T}_P$. Then, there exist in $\mathcal{T}(M)$ two fields of planes, one is the field of tangent planes T_P, and, the other is the field of vertical planes $\bar{T}_P$. A vector field is said to be *tangent*, if it belongs to the tangent plane T_P at each point P. A vector field is said to be *vertical*, if it belongs to the vertical plane $\bar{T}_P$ at each point P.

Let $\{B_b\} = \{B_1, B_2, \ldots, B_n\}$ be an ordered set of local tangent vector fields in a coordinate neighborhood U of M, such that

$$B_b x^a = \delta_b^a \;;\; i.e.\ B_b = \frac{\partial}{\partial x^b}$$

x^a being coordinates defined in U. We call the ordered set $\{B_b\}$ the *natural tangent frame* of the coordinates x^a in U. We denote by $B_b{}^h$ the components of the tangent vector fields B_b, that is,

$$B_b = B_b{}^h e_h.$$

Since the vertical plane $\bar{T}_P$ is m-dimensional, there exist in each coordinate neighborhood U, m vector fields $C_{n+1}, C_{n+2}, \ldots, C_N$ $(m = N - n)$ spanning the vertical plane $\bar{T}_P$ at each point P of U. The ordered set $\{C_y\} = \{C_{n+1}, C_{n+2}, \ldots C_N\}$ of such vertical vector fields is called a *vertical frame* in U. We denote by $C_y{}^h$ the components of the vertical vector fields C_y, that is,*) $C_y = C_y{}^h \mathbf{e}_h$.

If we define $B^a{}_h$ and $C^x{}_h$ by

$$\begin{aligned} &B^a{}_h B_b{}^h = \delta^a_b, \quad B^a{}_h C^h_y = 0, \\ &C^x{}_h B_b{}^h = 0, \quad C^x{}_h C_y{}^h = \delta^x_y. \end{aligned} \tag{4.1}$$

the two matrices

$$\begin{pmatrix} B^a{}_h \\ C^x{}_h \end{pmatrix}, \quad \begin{pmatrix} B_b{}^h \\ C_y{}^h \end{pmatrix}$$

are inverse to each other. We thus have

$$B_a{}^h B^a{}_i + C_x{}^h C^x{}_i = \delta^h_i \tag{4.2}$$

Let $V = v^a B_a$ be a tangent vector field. Then, v^a are the components of V considered as a vector field being tangent to the base space M. Since V has the components

$$V^h = B_a{}^h v^a$$

with respect to the N-frame $\{ \mathbf{e}_h \}$ in the hypertangent bundle $\mathcal{J}(M)$, the covariant derivative of V is given by

$$\begin{aligned} \nabla_c V^h &= \partial_c (B_b{}^h v^b) + \Gamma_c{}^h{}_i (B_b{}^i v^b) \\ &= (\partial_c B_b{}^h) v^b + B_b{}^h \partial_c v^b + \Gamma_{ci}{}^h B_b{}^i v^b \end{aligned} \tag{4.3}$$

in $\mathcal{J}(M)$. If we put

$$\nabla_c V^h = B_b^h P_c^{\ b} + C_x^h Q_c^{\ x}, \tag{4.4}$$

the coefficients $P_c{}^b$ and $Q_c{}^x$ are determined uniquely as follows :

$$\begin{aligned} &P_c^{\ b} = \partial_c v^b + \overset{\circ}{\Gamma}{}_c{}^b{}_a v^a, \\ &Q_c{}^x = (\partial_c B_b{}^h + \Gamma_c{}^h{}_i B_b{}^i) C^x{}_h v^b \end{aligned} \tag{4.5}$$

by means of (4.3), where functions $\overset{\circ}{\Gamma}{}_c{}^a{}_b$ are determined by

$$\overset{\circ}{\Gamma}{}_c{}^a{}_b = (\partial_c B_b{}^h + \Gamma_c^{\ h}{}_i B_b{}^i) B^a{}_h. \tag{4.6}$$

The functions $\overset{\circ}{\Gamma}{}_c^{\ a}{}_b$ defined above determine globally a linear connection in the base space M, which is called the *induced connection*. The induced connection $\overset{\circ}{\Gamma}{}_c^{\ a}{}_b$ is determined by the given hyperconnection $\Gamma_c^{\ h}{}_i$ and the choice of the vertical planes $\bar{T}_P$.

Denoting by $\overset{\circ}{\nabla}_b$ the operation of the covariant derivative with respect to the induced connection $\overset{\circ}{\Gamma}{}_c^{\ a}{}_b$ in the base space M, we obtain from (4.4) and (4.5)

$$\nabla_c V^h = B_b{}^h \overset{\circ}{\nabla}_c v^b + C_y{}^h H_{cb}{}^y, \tag{4.7}$$

---

* The indices x, y, z, u run over the range $\{n+1, n+2, \ldots, N.\}$

where we have put

$$\overset{\circ}{\nabla}_c v^b = \partial_c v^b + \Gamma_c{}^b{}_a v^a,$$

$$\begin{aligned} H_{cb}{}^x &= (\partial_c B_b{}^h + \Gamma_c{}^h{}_i B_b{}^i) C^x{}_h \\ &= (\partial_c B_b{}^h + \Gamma_c{}^h{}_l B_b{}^l - \overset{\circ}{\Gamma}_c{}^a{}_b B_a{}^h)\, C^x{}_h . \end{aligned} \tag{4.8}$$

It is easily seen that the quantity

$$\partial_c B_b{}^h + \Gamma_c{}^h{}_l B_b{}^l - \overset{\circ}{\Gamma}_c{}^a{}_b B_a{}^h$$

appearing in the right-hand side of (4.8) behaves as a vector field with respect to the index h and as a tensor field of (0,2)-type with respect to the indices c and b. Therefore, we know that the quantity $H_{cb}{}^x$ defined by (4.8) behaves as a vector field with respect to the index x by a change of the vertical frame $\{C_x\}$ and as a tensor field of (0, 2)-type with respect to the indices c and b.

Let $W = w^x C_x$ be a vertical vector field. Then, it has components

$$W^h = C_x{}^h w^x$$

with respect to the N-frame $\{e_h\}$. The covariant derivative of W is given by

$$\begin{aligned} \nabla_c W^h &= \partial_c (C_y{}^h w^y) + \Gamma_c{}^h{}_l (C_y{}^l\ w^y) \\ &= (\partial_c C_y{}^h) w^y + C_y{}^h \partial_c w^y + \Gamma_c{}^h{}_l C_y{}^l w^y . \end{aligned} \tag{4.9}$$

If we put

$$\nabla_c W^h = B_b{}^h R_c{}^b + C_x{}^h S_c{}^x, \tag{4.10}$$

the coefficients $R_c{}^b$, $S_c{}^x$ are determined uniquely as follows :

$$\begin{aligned} R_c{}^b &= (\partial_c C_y{}^h + \Gamma_c{}^h{}_l C_y{}^l) B^b{}_h w^y, \\ S_c{}^x &= \partial_c w^x + \overset{*}{\Gamma}_c{}^x{}_y w^y \end{aligned} \tag{4.11}$$

by means of (4.9), where the functions $\overset{*}{\Gamma}_c{}^x{}_y$ are given by

$$\overset{*}{\Gamma}_c{}^x{}_y = (\partial_c C_y{}^h + \Gamma_c{}^h{}_l C_y{}^l) C^x{}_h . \tag{4.12}$$

The functions $\overset{*}{\Gamma}_c{}^x{}_y$ defined above determine globally a linear connection in the vertical vector bundle, which is called the *vertical connection* induced from the given hyperconnection. The vertical connection is determined by the given hyperconnection $\Gamma_c{}^h{}_l$ and the choice of the vertical planes $\bar{T}_P$.

If we denote by $\overset{*}{\nabla}_c$ the operation of covariant derivation in the vertical vector bundle with respect to the vertical connection $\overset{*}{\Gamma}_c{}^x{}_y$, we obtain from (4.9) and (4.10)

$$\nabla_c W^h = B_b{}^h L_c{}^b{}_y w^y + C_y{}^h \overset{*}{\nabla}_c w^y, \tag{4.13}$$

where we have put

$$
\begin{aligned}
\overset{*}{\nabla}_c w^y &= \partial_c w^y + \overset{*}{\Gamma}{}_c{}^y{}_x\, w^x ,\\
L_{c.y}^{\;b} &= \left(\partial_c C_y^h + \Gamma_{c\,i}^{\,h} C_y^i \right) B^b_{\;h} \\
&= \left(\partial_c C_y^h + \Gamma_{c\,i}^{\,h} C_y^i - \overset{*}{\Gamma}{}_{c\,y}^{\,x} C_x^h \right) B^b_{\;h} .
\end{aligned}
\tag{4.14}
$$

It is easily verified that the quantity

$$\partial_c C_y^h + \Gamma_{c\,i}^{\,h} C_y^i - \overset{*}{\Gamma}{}_{c\,y}^{\,x} C_x^h$$

appearing in the right-hand side of (4.14) behaves as a covector field with respect to the index c and to the index y and as a vector field with respect to the index h. Then, the quantity $L_{c.y}^{\;b}$ behaves as a covector field with respect to the index y by a change of the vertical frame $\{C_x\}$ and as a tensor field of (1,1)-type with respect to the indices b and c.

Let V^h be an arbitrary vector field in the hypertangent bundle $\mathcal{J}(M)$. Then, it can be in a unique way expressed as

$$V^h = B_b^{\;h}\, v^b + C_y^{\;h}\, v^y ,$$

where the vector fields $B_b{}^h\, v^b$ and $C_y{}^h\, v^y$ are called respectively the *tangent part* and the *vertical part* of V^h. The covariant derivative of V^h is given by

$$\nabla_c V^h = B_b^{\;h}\left(\overset{\circ}{\nabla}_c v^b + L_{c.y}^{\;b}\, v^y \right) + C_y^{\;h}\left(H_{cb}^{\;\;h}\, v^b + \overset{*}{\nabla}_c\, v^y \right)$$

as a consequence of (4.7) and (4.13). Summing up, we get four quantities $\overset{\circ}{\Gamma}$, $\overset{*}{\Gamma}$, H, L which are determined by the given hyperconnection Γ and a choice of the field of vertical planes $\bar{T}_P$.

Conversely, we suppose that there is given a fixed field of vertical planes $\bar{T}_P$ in the hypertangent bundle $\mathcal{J}(M)$. Then, the set of all vertical planes $\bar{T}_P$ is a vertical vector bundle $\bar{T}(M)$. We assume moreover that a linear connection $\overset{\circ}{\Gamma}{}_c{}^a{}_b$ is given in the base space M and a linear connection $\overset{*}{\Gamma}{}_c{}^x{}_y$ is given in the vertical vector bundle $\bar{T}(M)$. Furthermore, we suppose that there are given two quantities $H_{cb}{}^x$ and $L_{c\cdot y}^{\;b}$ which have tensor property of corresponding types. Then, the quantities $\Lambda_c{}^h{}_i$ given by

$$
\begin{aligned}
\Lambda_{c\,b}^{\,a} &= \overset{\circ}{\Gamma}{}_{c\,b}^{\,a}, &\qquad \Lambda_{c\,b}^{\,x} &= H_{cb}^{\;\;x},\\
\Lambda_{c\,y}^{\,a} &= L_{c\cdot y}^{\;a} &\qquad \Lambda_{c\,y}^{\,x} &= \overset{*}{\Gamma}{}_{c\,y}^{\,x}
\end{aligned}
\tag{4.15}
$$

determine a hyperconnection in $\mathfrak{T}(M)$ with respect to an N-frame $\{\boldsymbol{B}_b, \mathbf{C}_y\}$ composed of a natural frame $\{\boldsymbol{B}_b\}$ and a vertical frame $\{\boldsymbol{C}_y\}$. Let $\{\mathbf{e}_h\}$ be an arbitrary N-frame in $\mathfrak{T}(M)$. Then, we may put

$$\boldsymbol{B}_b = B_b{}^h\,\mathbf{e}_h, \qquad \boldsymbol{C}_y = C_y{}^h\,\mathbf{e}_h;$$
$$\mathbf{e}_h = B^a{}_h\,\boldsymbol{B}_a + C^x{}_h\,\boldsymbol{C}_x, \tag{4.16}$$

which give the law of transformation between two N-frames $\{\mathbf{e}_h\}$ and $\{\boldsymbol{B}_b, \boldsymbol{C}_y\}$. By making use of (4.15) and (4.16), we obtain the components $\Gamma_c{}^h{}_i$ of the hyperconnection defined by (4.15) as follows :

$$\begin{aligned}\Gamma_c{}^h{}_i = {}&\left(\partial_c B^a{}_i + \overset{\circ}{\Gamma}{}_c{}^a{}_b B^b{}_i + L_{c.y}^{\ a} C^y{}_i\right) B_a{}^h \\ &+ \left(\partial_c C^x{}_i + \overset{*}{\Gamma}{}_c{}^x{}_y + H_{cb}{}^x B^b{}_i\right) C_x{}^h\end{aligned} \tag{4.17}$$

or,

$$\begin{aligned}\Gamma_c{}^h{}_i = {}&\left(H_{cb}{}^x C_x{}^h - \partial_c B_b{}^h + \overset{\circ}{\Gamma}{}_c{}^a{}_b B_a{}^h\right) B^b{}_i \\ &+ \left(L_{c.y}^{\ b} B_b{}^h - \partial_c C_y{}^h + \overset{*}{\Gamma}{}_c{}^x{}_y C_x{}^h\right) C^y{}_i,\end{aligned} \tag{4.18}$$

with respect to the N-frame $\{\mathbf{e}_h\}$, taking account of (3.5). This means that a hyperconnection is completely determined by giving four quantities $\overset{\circ}{\Gamma}$, $\overset{*}{\Gamma}$, H, L and by a choice of the vertical vector bundle $\bar{T}(M)$.

5. Curvature and torsion.

Let $\mathfrak{T}(M)$ be an N-dimensional hypertangent bundle over a base space M. There exists then a fibre bundle $a(M)$ which is associated with $\mathfrak{T}(M)$, has the N-dimensional affine space as its fibre space and the group of all affine motions as its structure group. The bundle $a(M)$ is called the N-dimensional *hyperaffine bundle* associated with $\mathfrak{T}(M)$.

Let $\Gamma_c{}^h{}_i$ be a hyperconnection in a hypertangent bundle $\mathfrak{T}(M)$. Using the components $B_b{}^h$ of the tangent vector fields $\boldsymbol{B}_b$ which form the natural frame $\{\boldsymbol{B}_b\}$, we define N forms ω^h by

$$\omega^h = B_b{}^h dx^b. \tag{5.1}$$

If we now consider a matrix π of differential forms

$$\pi = \begin{pmatrix} 0 & \omega^h \\ 0 & \omega_i^h \end{pmatrix}, \tag{5.2}$$

the forms ω_i^h being defined by

$$\omega_i^h = \Gamma_c{}^h{}_i dx^c, \tag{5.3}$$

we see easily that the matrices defined in each coordinate neighborhood determine globally an infinitesimal connection in the principal bundle associated with the hyperaffine bundle $a(M)$ which is associated with $\mathcal{J}(M)$. The infinitesimal connection π thus defined is called the *affine connection* of $\mathcal{J}(M)$ corresponding to the hyperconnection $\Gamma^{\ h}_{c\ i}$

The curvature form Ω of the affine connection corresponding to $\Gamma^{\ h}_{c\ i}$ is by definition

$$\Omega = d\pi - \pi\wedge\pi, \tag{5.4}$$

which is expressed as follows :

$$\Omega = \begin{pmatrix} 0 & \Omega^h \\ 0 & \Omega^h_i \end{pmatrix} \tag{5.5}$$

because of (5.2), where

$$\begin{aligned} \Omega^h &= d\omega^h - \omega^i\wedge\omega^h_i, \\ \Omega^h_i &= d\omega^h_i - \omega^k_i\wedge\omega^h_k. \end{aligned} \tag{5.6}$$

On putting

$$\begin{aligned} \Omega^h &= S_{cb}{}^h dx^c\wedge dx^b, & S_{cb}{}^h + S_{bc}{}^h &= 0, \\ \Omega^h_i &= \tfrac{1}{2}K_{cbi}{}^h dx^c\wedge dx^b, & K_{cbi}{}^h + K_{bci}{}^h &= 0, \end{aligned} \tag{5.7}$$

we obtain from (5.1), (5.3) and (5.6)

$$\begin{aligned} S_{cb}^{\ h} &= \tfrac{1}{2}\left(\partial_c B_b{}^h + \Gamma^{\ h}_{ci}\, B_b{}^i\right) - \left(\partial_b B_c{}^h + \Gamma^{\ h}_{b\ i} B_c{}^i\right), \\ K_{cbi}{}^h &= \partial_c \Gamma^{\ h}_{b\ i} - \partial_b \Gamma^{\ h}_{c\ i} + \Gamma^{\ h}_{c\ k}\Gamma^{\ k}_{b\ i} - \Gamma^{\ h}_{b\ k}\Gamma^{\ k}_{c\ i} \end{aligned} \tag{5.8}$$

The coefficients $S_{cb}{}^h$ of Ω^h form a tensor field which is of (0, 2)-type with respect to the indices c, b and a vector field with respect to the index h. The coefficients $K_{cbi}{}^h$ of Ω^h_i form a tensor field which is of (0, 2)-type with respect to the indices c, b and of (1, 1)-type with respect to the indices h and i. The tensor field $S_{cb}{}^h$ is called the *torsion tensor* of the hyperconnection $\Gamma^{\ h}_{c\ i}$. The tensor field $K_{cbi}{}^h$ is called the *curvature tensor* of the hyperconnection $\Gamma^{\ h}_{c\ i}$.

Taking account of (4.6), we see that the torsion tensor $\mathring{S}_{cb}{}^h$ of the induced connection $\mathring{\Gamma}^{\ a}_{c\ b}$ is given by

$$\begin{aligned} \mathring{S}_{cb}{}^a &= \tfrac{1}{2}\left(\mathring{\Gamma}^{\ a}_{c\ b} - \mathring{\Gamma}^{\ a}_{b\ c}\right) \\ &= \tfrac{1}{2}\left[\left(\partial_c B_b{}^h + \mathring{\Gamma}^{\ h}_{c\ i} B_b{}^i\right) - \left(\partial_b B_c + \mathring{\Gamma}^{\ h}_{b\ i} B_c{}^i\right)\right] B^a{}_h, \end{aligned}$$

which implies together with (5.8)

$$\overset{\circ}{S}_{cb}{}^{h} = S_{cb}{}^{a} B_{a}{}^{h}. \tag{5.9}$$

This gives a relation between the torsion tensors of the hyperconnection and its induced connection.

If the torsion tensor $S_{cb}{}^{h}$ of a hyperconnection $\Gamma_{c}{}^{h}{}_{i}$ vanishes identically, the hyperconnection $\Gamma_{c}{}^{h}{}_{i}$ is said to *have no torsion.*

THEOREM 5.1 *If a hyperconnection has no torsion, then its induced connection is symmetric.*

Let $S_{cb}{}^{h}$ be the torsion tensor of a hyperconnection $\Gamma_{c}{}^{h}{}_{i}$. Then, we can define another hyperconnection $\Lambda_{c}{}^{h}{}_{i}$ by

$$\Lambda_{c}{}^{h}{}_{i} = \Gamma_{c}{}^{h}{}_{i} - S_{cb}{}^{h} B^{b}{}_{i}$$

It is now easily verified that the torsion tensor of the new hyperconnection $\Lambda_{c}{}^{h}{}_{i}$ vanishes identically. We have thus

THEOREM 5.2. *In any hypertangent bundle, there exists always a hyperconnection which has no torsion.*

It follows from (4.8) and (5.8)

$$H_{cb}{}^{x} - H_{bc}{}^{x} = 2S_{cb}{}^{h} C^{x}{}_{h}.$$

Thus, if we call the tensor

$$S_{cb}{}^{x} = S_{cb}{}^{h} C^{x}{}_{h} = \tfrac{1}{2}(H_{cb}{}^{x} - H_{bc}{}^{x}) \tag{5.10}$$

the *vertical torsion* of the hyperconnection $\Gamma_{c}{}^{h}{}_{i}$, then we obtain from (5.9) and (5.10)

$$\begin{aligned} S_{cb}{}^{h} &= S_{cb}{}^{a} B_{a}{}^{h} + S_{cb}{}^{x} C_{x}{}^{h} \\ &= \tfrac{1}{2}(\overset{\circ}{\Gamma}_{c}{}^{a}{}_{b} - \overset{\circ}{\Gamma}_{b}{}^{a}{}_{c}) B_{a}{}^{h} + \tfrac{1}{2}(H_{cb}{}^{x} - H_{bc}{}^{x}) C_{x}{}^{h}, \end{aligned}$$

which implies

THEOREM 5.3. *A hyperconnection has no torsion if and only if the induced connection is symmetric and $H_{cb}{}^{x}$ is symmetric with respect to the lower indices c and b.*

LEMMA 5.4. *The tensor $H_{cb}{}^{x}$ is symmetric with respect to the lower indices c and b, if and only if the vertical torsion tensor vanishes identically. When the hyperconnection has no torsion, the tensor $H_{cb}{}^{x}$ is symmetric with respect to the lower indices* c *and b.*

6. **Identities containing curvature tensors and torsion tensors.**

We shall first introduce a new covariant differentiation. We take a tensor field $T_{b}{}^{ah}$ for example, which is of (1, 1)-type with respect to the indices *a*, *b* and of (1, 0)-type with respect to the index *h*. We now define the covariant derivative of $T_{b}{}^{ah}$ by

$$\nabla_{c} T_{b}{}^{ah} = \partial_{c} T_{b}{}^{ah} + \Gamma_{c}{}^{h}{}_{i} T_{b}{}^{ai} + \overset{\circ}{\Gamma}_{c}{}^{a}{}_{d} T_{b}{}^{dh} - \overset{\circ}{\Gamma}_{c}{}^{d}{}_{b} T_{d}{}^{ah}, \tag{6.1}$$

which is a tensor field of (1, 2)-type with respect to the indices c, b, a and of (1, 0)-type with respect to the index h, $\Gamma_c{}^h{}_i$ and $\overset{\circ}{\Gamma}_c{}^a{}_b$ being the hyperconnection and its induced connection respectively. In a similar way we can define the covariant derivative of tensor fields of any mixed type of this kind.

We shall next define the covariant derivative of a tensor field $T_i{}^{hx}$, for example, which is of (1, 1)-type with respect to the indices h, i and of (1, 0)-type with respect to the index x. That is,

$$\nabla_c T_i{}^{hx} = \partial_c T_i{}^{hx} + \Gamma_c{}^h{}_k T_i{}^{kx} - \Gamma_c{}^k{}_i T_k{}^{hx} + \overset{*}{\Gamma}_c{}^x{}_y T_i{}^{hy}, \tag{6.2}$$

which is a tensor field of (1, 1)-type with respect to the indices h, i, of (0, 1)-type with respect to the index c, and of (1, 0)-type with respect to the index x, $\Gamma_c{}^h{}_i$ and $\overset{*}{\Gamma}_c{}^x{}_y$ being the hyperconnection and the vertical connection respectively. In a similar way, we can define the covariant derivatives of tensor fields of any mixed type of this kind.

We have the following identity because of (5.6)

$$\omega^i \wedge \Omega_i^h = d\Omega^h + \Omega^i \wedge \omega_i^h .$$

Substituting (5.7) in both the sides of this equation, we obtain

$$\begin{aligned} B_d{}^i K_{cbi}{}^h + B_c{}^i K_{bdi}{}^h + B_b{}^i K_{dci}{}^h & \\ = 2(\nabla_d S_{cb}{}^h + \nabla_c S_{bd}{}^h + \nabla_b S_{dc}{}^h) & \\ + 4(S_{dc}{}^a S_{ab}{}^h + S_{cb}{}^a S_{ad}{}^h + S_{bd}{}^a S_{ac}{}^h), & \end{aligned} \tag{6.3}$$

which corresponds to the first Bianchi identity.

It follows easily from (5. 6) that the identity

$$d\Omega_i^h + \Omega_i^k \wedge \omega_k^h + \omega_i^k \wedge \Omega_k^h = 0$$

is valid. Substituting (5.7) in the left-hand side of this equation, we obtain

$$\begin{aligned} \nabla_d K_{cbi}{}^h + \nabla_c K_{bdi}{}^h + \nabla_b K_{dci}{}^h & \\ + 2(S_{dc}{}^a K_{abi}{}^h + S_{cb}{}^a K_{adi}{}^h + S_{bd}{}^a K_{aci}{}^h) = 0, & \end{aligned} \tag{6.4}$$

which corresponds to the second Bianchi identity.

When the hyperconnection $\Gamma_c{}^h{}_i$ has no torsion, (6.3) and (6.4) are reduced respectively to

$$B_d{}^i K_{cbi}{}^h + B_c{}^i K_{bdi}{}^h + B_b{}^i K_{dci}{}^h = 0, \tag{6.5}$$

and

$$\nabla_d K_{cbi}{}^h + \nabla_c K_{bdi}{}^h + \nabla_b K_{dci}{}^h = 0. \tag{6.6}$$

When the induced connection $\overset{\circ}{\Gamma}_c{}^a{}_b$ is symmetric, (6.6) is valid.

To find the relation between the curvature tensors of the hyperconnection $\Gamma_c{}^h{}_i$ and that of the induced connection $\overset{o}{\Gamma}_c{}^a{}_b$, we shall use the identity

$$\nabla_d \nabla_c B_b{}^h - \nabla_c \nabla_d B_b{}^h = B_b{}^i K_{dci}{}^h - B_a{}^h R_{dcb}{}^a, \tag{6.7}$$

where $R_{dcb}{}^a$ denotes the curvature tensor of the induced connection $\overset{o}{\Gamma}_c{}^a{}_b$ and is given by

$$R_{dcb}{}^a = \partial_d \overset{o}{\Gamma}_c{}^a{}_b - \partial_c \overset{o}{\Gamma}_d{}^a{}_b + \overset{o}{\Gamma}_d{}^a{}_e \overset{o}{\Gamma}_c{}^e{}_b - \overset{o}{\Gamma}_c{}^a{}_e \overset{o}{\Gamma}_d{}^e{}_b.$$

If we follow the notation introduced by (6.1), we have, taking account of (4.6)

$$(\nabla_c B_b{}^h) B^a{}_h = 0,$$

and, taking account of (4.8),

$$(\nabla_c B_b{}^h)\ C^x{}_h = H_{cb}{}^x.$$

Then, it follows from the equations obtained above

$$\nabla_c B_b{}^h = H_{cb}{}^x C_x{}^h \tag{6.8}$$

Following the notation introduced by (6.2), we have from (4.12)

$$\left(\nabla_c C_y{}^h\right) C^x{}_h = 0,$$

and from (4.14)

$$\left(\nabla_c C_y{}^h\right) B^b{}_h = L_{c.y}^{\ b}.$$

Then, it follows from the equations now obtained

$$\nabla_c C_y{}^h = L_{c.y}^{\ b} B_b{}^h. \tag{6.9}$$

Substituting (6.8) in the left-hand side of (6.7), and taking account of (6.9), we obtain

$$\left(\nabla_d H_{cb}{}^x - \nabla_c H_{db}{}^x\right) C_x{}^h + \left(L_{d.y}^{\ a} H_{cb}{}^y - L_{c.y}^{\ a} H_{db}{}^y\right) B_a{}^h$$
$$= K_{dci}{}^h B_b{}^i - R_{dcb}{}^a B_a{}^h,$$

which implies

$$\begin{aligned} K_{dcb}{}^a &= R_{dcb}{}^a + \left(L_{d.y}^{\ a} H_{cb}{}^y - L_{c.y}^{\ a} H_{db}{}^y\right), \\ K_{dcb}{}^x &= \nabla_d H_{cb}{}^x - \nabla_c H_{db}{}^x, \end{aligned} \tag{6.10}$$

where we have put

$$K_{dcb}{}^a = K_{dci}{}^h B_b{}^i B^a{}_h,$$

$$K_{dcb}{}^x = K_{dci}{}^h B_b{}^i C^x{}_h.$$

To obtain relations between the curvature tensors of the hyperconnection $\Gamma_{c\,i}^{\,h}$ and that of the vertical connection $\overset{*}{\Gamma}{}_{c\,y}^{\,x}$, we shall use the identity

$$\nabla_d \nabla_c C_y{}^h - \nabla_c \nabla_d C_y{}^h = C_y K_{dci}{}^h - C_z{}^h \overset{*}{R}_{dcy}{}^z, \tag{6.11}$$

where $\overset{*}{R}_{dcy}{}^z$ denotes the curvature tensor of the vertical connection $\overset{*}{\Gamma}{}_{c\,y}^{\,x}$ and is given by

$$\overset{*}{R}_{dcy}{}^x = \partial_d \overset{*}{\Gamma}{}_{c\,y}^{\,x} - \partial_c \overset{*}{\Gamma}{}_{dy}^{\,x} + \overset{*}{\Gamma}{}_{d\,z}^{\,x} \overset{*}{\Gamma}{}_{c\,y}^{\,z} - \overset{*}{\Gamma}{}_{c\,z}^{\,x} \overset{*}{\Gamma}{}_{d\,y}^{\,z}.$$

Substituting (6.9) in the left-hand side of (6.11), and taking account of (6.8), we obtain

$$\left(\nabla_d L_{c.y}^{\,b} - \nabla_c L_{d.y}^{\,b}\right) B_b{}^h + \left(H_{db}{}^x L_{c.y}^{\,b} - H_{cb}{}^x L_{d.y}^{\,b}\right) C_x{}^h$$
$$= C_y{}^i K_{dci}{}^h - C_x{}^h \overset{*}{R}_{dcy}{}^x,$$

which implies

$$\begin{aligned} K_{dcy}{}^x &= \overset{*}{R}_{dcy}{}^x + \left(H_{dc}{}^x L_{c.y}^{\,b} - H_{cb}{}^x L_{d.y}^{\,b}\right), \\ K_{dcy}{}^a &= \nabla_d L_{c.y}^{\,a} - \nabla_c L_{d.y}^{\,a}, \end{aligned} \tag{6.12}$$

where we have put

$$K_{dcy}{}^x = K_{dci}\ C_y{}^i C^x{}_h,$$
$$K_{dcy}{}^a = K_{dci}{}^h C_y{}^i B^a{}_h.$$

We have now the following expression of the curvature tensor $K_{dci}{}^h$ of the hyperconnection $\Gamma_{c\,i}^{\,h}$:

$$\begin{aligned} K_{dci}{}^h = &\left[R_{dcb}{}^a + \left(L_{d.x}^{\,a} H_{cb}{}^x - L_{c.x}^{\,a} H_{db}{}^x\right) B^b{}_i B_a{}^h\right. \\ &+ \left[\overset{*}{R}_{dcy}{}^x + \left(H_{db}{}^x L_{c.y}^{\,b} - H_{cb}{}^x L_{d.y}^{\,b}\right)\right] C^y{}_i C_x{}^h \\ &+ \left[\nabla_d H_{cb}{}^x - \nabla_c H_{db}{}^x\right] B^b{}_i C_x{}^h \\ &+ \left[\nabla_d L_{c.y}^{\,b} - \nabla_c L_{d.y}^{\,b}\right] C^y{}_i B_b{}^h \end{aligned} \tag{6.13}$$

as a consequence of (6.10) and (6.12).

7. **Paths and autoparallel curves.**

In the base space M, the path with respect to the induced connection $\overset{\circ}{\Gamma}{}_{c\,b}^{\,a}$ is called briefly the *path of the base space*. Then, a path $x^a = x^a(t)$ of the base space is defined by the differential equation

$$\frac{\delta^2 x^a}{dt^2} = \frac{d^2 x^a}{dt^2} + \overset{\circ}{\Gamma}{}_{c\,b}^{\,a} \frac{dx^c}{dt} \frac{dx^b}{dt} = 0 \tag{7.1}$$

with an affine parameter t. The tangent vector of the path is expressed as follows :

$$V^h = \frac{dx^a}{dt} B_a{}^h$$

in the hypertangent bundle $\mathcal{J}(M)$. If we denote by

$$\frac{\delta V^h}{dt} = \frac{dV^h}{dt} + \Gamma_c{}^h{}_i \frac{dx^c}{dt} V^i$$

the covariant derivative of the vector field V^h along the path itself, we have

$$\frac{\delta V^h}{dt} = \frac{\delta^2 x^a}{dt^2} B_a{}^h + \nabla_c B_b{}^h \frac{dx^c}{dt}\frac{dx^b}{dt}.$$

Because of (6.8) and (7.1), this equation is reduced to

$$\frac{\delta V^h}{dt} = \left(H_{cb}{}^x \frac{dx^c}{dt}\frac{dx^b}{dt}\right) C_x{}^h, \tag{7.2}$$

which implies

THEOREM 7.1. *The tangent vector* $\frac{dx^a}{dt} B_a{}^h$ *of a path* $x^a = x^a(t)$ *is parallel along the path itself, if and only if*

$$H_{cb}{}^x \frac{dx^c}{dt}\frac{dx^b}{dt} = 0\ .$$

We consider a curve $x^a = x^a(r)$ in the base space M. We assume that there exists a vector field V^h along the curve such that its tangent part is tangent to the curve. Then, V^h can be written as follows :

$$V^h = \frac{dx^a}{dr} B_a{}^h + v^x C_x{}^h, \tag{7.3}$$

where $v^x C_x{}^h$ is a certain vertical vector field along the curve. We suppose moreover the vector field V^h to be parallel to itself in $\mathcal{J}(M)$, that is,

$$\frac{\delta V^h}{dr} = \alpha(r)\, V^h, \tag{7.4}$$

where $\alpha(r)$ is a certain function along the curve. A curve lying on the base space M and satisfying the above mentioned conditions is called an *autoparallel curve.*

Substituting (7.3) in both the sides of (7.4), we obtain

$$\begin{aligned}\left(\frac{\delta^2 x^a}{dr^2} + L_{c\cdot y}{}^a \frac{dx^c}{dr} v^y\right) B_a{}^h + \left(\frac{\delta v^x}{dr} + H_{cb}{}^x \frac{dx^c}{dr}\frac{dx^b}{dr}\right) C_x{}^h & \\ = \alpha V^h & \\ = \alpha\left(\frac{dx^a}{dr} B_a{}^h + v^x C_x{}^h\right), &\end{aligned}$$

where we have put

$$\frac{\delta v^x}{dr}=\frac{dv^x}{dr}+\overset{*}{\Gamma}{}_{c\,y}^{\;\;x}\frac{dx^c}{dr}v^y.$$

It follows then

$$\frac{\delta^2 x^a}{dr^2}+L_{c\cdot y}^{\;a}\frac{dx^c}{dr}v^y=\alpha\frac{dx^a}{dr},$$
$$\frac{\delta v^x}{dr}+H_{cb}{}^x\frac{dx^c}{dr}\frac{dx^b}{dr}=\alpha v^x. \tag{7.5}$$

These are the equations determining the autoparallel curves.

If an autoparallel curve $x^a=x^a(r)$ satisfies

$$H_{cb}{}^x\frac{dx^c}{dr}\frac{dx^b}{dr}=0,$$

the second equation of (7.5) is reduced to

$$\frac{dv^x}{dr}=\alpha\, v^x,$$

which implies that this autoparallel curve is a path of the base space. We have thus

THEOREM 7.2. *An autoparallel curve $x^a=x^a(r)$ satisfying*

$$H_{cb}{}^x\frac{dx^c}{dr}\frac{dx^b}{dr}=0$$

is a path of the base space. If a path $x^a=x^a(r)$ satisfies the above condition it is an autoparallel curve.

8. **Conditions for being geodesic.**

If, in a hypertangnet bundle with a hyperconnection $\Gamma_{c\,i}^{\;h}$ the tangent spaces T_p of the base space M are parallel to themselves, then the base space M is said to be *parallel.* The base space M is parallel if and only if

$$\left(\nabla_c B_b{}^h\right) C_h{}^x=0,$$

which implies together with (4.8)

$$H_{cb}{}^x=0.$$

Therefore, we have

THEOREM 8.1. *The base space M is parallel if and only if*

$$H_{cb}{}^x=0,$$

We shall now suppose that for any path $x^a = x^a(t)$ of the base space M its tangent vectors $\frac{dx^a}{dt} B_a{}^h$ in the hypertangent bundle $\mathcal{J}(M)$ are parallel to themselves along the path. In such a case, the base space M is said to be *totally geodesic.* We have now easily from Theorem 7.1

THEOREM 8.2. *The base space M is totally geodesic if and only if*

$$H_{cb}{}^x + H_{bc}{}^x = 0.$$

Evidently, the base space is totally geodesic if it is parallel. Theorems 7.2 and 8.2 imply

THEOREM 8.3. *When the base space M is totally geodesic, a curve in the base space is autoparallel if and only if it is a path.*

We have moreover from Theorems 8.1, 8.2 and Lemma 5.4

THEOREM 8.4. *When the hyperconnection has no torsion, or, when it has no vertical torsion, the base space M is totally geodesic if and only if it is parallel.*

9. Locally flat hyperconnection and its example.

If, in a hypertangent bundle $\mathcal{J}(M)$, a hyperconnection has no torsion and its curvature tensor vanishes identically, the hyperconnection is said to be *locally flat.* We shall now show an example of the hypertangent bundle admitting a locally flat hyperconnection.

Let A_N be an affine space of N dimensions. We consider now a submanifold M immersed in A_N, that is, M is a submanifold of A_N which may have self-intersection at some points. Let $T(A_N)$ be the tangent bundle of A_N and denote by

$$i : M \longrightarrow A_N$$

the injection mapping of M into A_N. Then, the fibre bundle $\mathcal{J}(M)$ induced from $T(A_N)$ by the injection mapping $i : M \longrightarrow A_N$ is an N-dimensional hypertangent bundle over the submanifold M [7]. In fact, it is well-known that $\mathcal{J}(M)$ contains the tangent bundle $T(M)$ of M in the natural way as a subbundle. The vector bundle $\mathcal{J}(M)$ is called the *natural hypertangent bundle* over the submanifold M. On the other hand, the natural hypertangent bundle $\mathcal{J}(M)$ can be immersed natually into the tangent bundle $T(A_N)$. We denote by

$$j : \mathcal{J}(M) \longrightarrow T(A_N)$$

this immersion mapping.

The affine space A_N admits naturally an affine connection $\hat{\Gamma}_c{}^h{}_i$ which is locally flat; that is, $\hat{\Gamma}_c{}^h{}_i$ has no torsion and its curvature tensor vanishes identically. The locally flat affine connection $\hat{\Gamma}_c{}^h{}_i$ determines uniquely in the tangent bundle $T(A_N)$ a linear connection

$\hat{\omega}$, which is locally flat. Then, the mapping j: $\mathcal{T}(M) \longrightarrow T(A_N)$ induces a linear connection ω in $\mathcal{T}(M)$. The connection ω is a hyperconnection in $\mathcal{T}(M)$ and obviously locally flat. It is called the *natural hyperconnection* in the natural hypertangent bundle $\mathcal{T}(M)$ of the submanifold M immersed in A_N. Thus, we have a typical example of hyperconnection.

10. **Realization of a hypertangent bundle admitting a locally flat hyperconnection.**

Let $\mathcal{T}(M)$ and $\mathcal{T}(\tilde{M})$ be two N-dimensional hypertangent bundles over differentiable manifolds M and $\tilde{M}$ respectively. We suppose now that there exists a differentiable bundle-isomorphism

$$\phi : \mathcal{T}(M) \longrightarrow \mathcal{T}(\tilde{M}).$$

That is to say, the mapping ϕ is a differentiable homeomorphism of $\mathcal{T}(M)$ onto $\mathcal{T}(\tilde{M})$ and gives an equivalence between two bundle structures of $\mathcal{T}(M)$ and $\mathcal{T}(\tilde{M})$ [7]. Then, the bundle-isomorphism ϕ induces naturally a differentiable homeomorphism $\phi_0 : M \rightarrow \tilde{M}$ of the base space M onto $\tilde{M}$. If there are given two hyperconnections ω and $\tilde{\omega}$ respectively in $\mathcal{T}(M)$ and $\mathcal{T}(\tilde{M})$ such that ω is equal to the connection induced from $\tilde{\omega}$ by the mapping ϕ, we say that the pair $\{\mathcal{T}(M), \omega\}$ of the hypertangent bundle $\mathcal{T}(M)$ and a hyperconnection ω is equivalent to the pair $\{\mathcal{T}(\tilde{M}), \tilde{\omega}\}$. We shall here prove the following lemma.

LEMMA 10.1. *Let U be a cubic coordinate neighborhood of n dimensions and $\mathcal{T}(U)$ be an N-dimensional hypertangent bundle with a hyperconnection ω, which is locally flat. Then, if U is sufficiently small, there exists an n-dimensional submanifold $\tilde{U}$ (imbedded) in the N-dimensional affine space A_N such that the pair $\{\mathcal{T}(U), \omega\}$ is equivalent to the pair $\{\mathcal{T}(\tilde{U}), \tilde{\omega}\}$ of the natural hypertangent bundle over $\tilde{U}$ and the natural hyperconnection $\tilde{\omega}$ in $\mathcal{T}(\tilde{U})$. For the given $\mathcal{T}(U)$ and the locally flat hyperconnection ω, the natural hypertangent bundle $\mathcal{T}(\tilde{U})$ is determined up to affine transformations of A_N.*

PROOF. We take an N-frame $\{\mathbf{e}_h\}$ in $\mathcal{T}(U)$ and denote by $\Gamma_c{}^h{}_i$ the components of the given hyperconnection ω with respect to the N-frame $\{\mathbf{e}_h\}$. Then, $\Gamma_c{}^h{}_i$ are functions of coordinates x^a defined in U. Let $B_b{}^h$ be the components of tangent vectors $\boldsymbol{B}_b$ forming the natural frame of the coordinates x^a in U. Then, $B_b{}^h$ are functions of coordinates x^a.

In the affine space A_N, denoting by ξ^h a system of Cartesian coordinates, we consider the following partial differential equations

$$\begin{aligned} \frac{\partial \xi^h}{\partial x^b} &= B_b{}^l(x)\, V_l{}^h, \\ \frac{\partial V_l{}^h}{\partial x^b} &= \Gamma_b{}^k{}_l(x)\, V_k{}^h \end{aligned} \tag{10.1}$$

with unknown functions ξ^h and $V_l{}^h$, where $B_b{}^h(x)$ and $\Gamma_b{}^h{}_l(x)$ are equal to the corresponding functions in $\mathcal{T}(U)$ with respect to $\{\mathbf{e}_h\}$. Hence, $B_b{}^h(x)$ and $\Gamma_b{}^k{}_l(x)$ are given functions of the independent variable x^a. The partial differential equations (10.1) are defined in U, which is identified with a cubic domain in the space of n variables x^a.

Since the hyperconnection $\Gamma_{c\,i}^{\,h}$ is locally flal in $\mathcal{J}(U)$, we have from (5.8)

$$\left(\partial_c B_b^{\,h}+\Gamma_{c\,i}^{\,h}\, B_b^{\,i}\right)-\left(\partial_b B_c^{\,h}+\Gamma_{b\,i}^{\,h}\, B_c^{\,i}\right)=0 \tag{10.2}$$

as a consequence of

$$S_{cb}^{\;\;h}=0,$$

and

$$\partial_c\Gamma_{b\,i}^{\,h}-\partial_b\Gamma_{c\,i}^{\,h}+\Gamma_{c\,k}^{\,h}\Gamma_{b\,i}^{\,k}-\Gamma_{b\,k}^{\,h}\,\Gamma_{c\,i}^{\,k}=0 \tag{10.3}$$

as a direct consequence of

$$K_{cbi}^{\;\;\;h}=0.$$

We shall verify that the partial differential equation (10.1) is completely integrable. If we put

$$\begin{aligned} F_b^{\,h}\,(x,\, V)&=B_b^{\,i}\,(x)\; V_i^{\,h},\\ F_{bi}^{\;\;h}\,(x,\, V)&=\Gamma_{b\,i}^{\,k}\,(x)\; V_k^{\,h}, \end{aligned} \tag{10.4}$$

(10.1) is reduced to

$$\frac{\partial \xi^h}{\partial x^b}=F_b^{\,h},\quad \frac{\partial V_i^{\,h}}{\partial x^b}=F_{bi}^{\;\;h} \tag{10.5}$$

Differentiating $F_b^{\,h}$ and $F_{bi}^{\;\;h}$ by x^c, and taking account of (10.4), we obtain

$$\partial_c F_b^{\,h}=(\partial_c B_b^{\,k}+\Gamma_{c\,j}^{\,k}\, B_b^{\,j})\; V_k^{\,h},$$

$$\partial_c F_{bi}^{\;\;h}=(\partial_c \Gamma_{b\,i}^{\,k}+\Gamma_{c\,j}^{\,k}\,\Gamma_{b\,i}^{\,j})\; V_k^{\,h}.$$

Thus, we have

$$\partial_c F_b^{\,h}-\partial_b F_c^{\,h}=0,$$

$$\partial_c F_{bi}^{\;\;h}-\partial_b F_{ci}^{\;\;h}=0$$

because of (10.2) and (10.3) respectively. Therefore, the differential equation (10.4) is completely integrable and consequently the differential equation (10.1) is also so.

The differential equation (10.1) being completely integrable, if we take arbitrarily a set $\left\{\overset{\circ}{\xi}{}^h,\ \overset{\circ}{V}{}_i^{\,h}\right\}$ of constants in correspondence with the unknown functions $\left\{\xi^h,\ V_i^{\,h}\right\}$, there exists a solution $\left\{\xi^h\,(x),\ V_i^{\,h}\,(x)\right\}$ of (10.1) in U such that

$$\xi^h\,(\overset{\circ}{x})=\overset{\circ}{\xi}{}^h,\quad V_i^{\,h}\,(\overset{\circ}{x})=\overset{\circ}{V}{}_i^{\,h} \tag{10.6}$$

at a given point $\overset{\circ}{x}{}^a$ belonging to U, For a solution $\left\{\xi^h, V_i{}^h\right\}$ of (10.1), if we denote by $\triangle$ the determinant $\left|V_i{}^h\right|$ of the matrix $(V_i{}^h)$, it is easily seen that the function $\triangle$ is a solution of the differential equation

$$\frac{\partial\triangle}{\partial x^b}=\left(\Gamma_b{}^h{}_h\right)\triangle, \tag{10.7}$$

which is linear and homogeneous with respect to the unknown function $\triangle$. Hence, if a solution $\triangle(x)$ of (10.7) does not vanish at a point x^a of U, we have $\triangle(x)\neq 0$ at each point x^a of U. Therefore, if we take a set $\left\{\overset{\circ}{\xi}{}^h, \overset{\circ}{V}_i{}^h\right\}$ of initial values such that $\overset{\circ}{\triangle}=\left|\overset{\circ}{V}_i{}^h\right|\neq 0$, for the solution $\left\{\xi^h(x), V_i{}^h(x)\right\}$ of (10.1) satisfying the initial condition (10.7) we obtain

$$\triangle(x)=\left|V_i{}^h(x)\right|\neq 0 \tag{10.8}$$

at each point x^a of U. Consequently, there exists a solution $\left\{\xi^h(x), V_i{}^h(x)\right\}$ of (10.1) satisfying (10.8). By making use of such a solution, we can define in the affine space A_N a submanifold $\tilde{U}$ by the equation

$$\xi^h=\xi^h(x)$$

and a N-frame $\{V_i\}$ along U formed from N vector fields

$$V_1,\ V_2,\ \ldots\ldots\ldots V_N$$

with components

$$V_1{}^h(x),\ V_2{}^h(x), \ldots\ldots V_N{}^h(x)$$

respectively, because these N vector fields are linearly independent by virtue of (10.8). It is easily seen that the submanifold $\tilde{U}$ is differentiablly homeomorphic to U if U is sufficiently small. If in the natural hypertangent bundle $\mathcal{J}(\tilde{U})$ over $\tilde{U}$ we now introduce a hyperconnection $\tilde{\omega}$ having components $\tilde{\Gamma}_c{}^h{}_i(x)$ with respect to the N-frame $\{V_b\}$ above constructed, $\tilde{\omega}$ is the natural hyperconnection in $\mathcal{J}(\tilde{U})$. Then, we can see easily that the given pair $\{\mathcal{J}(U), \omega\}$ is equivalent to the pair $\{\mathcal{J}(\tilde{U}), \tilde{\omega}\}$ in the natural way. It is easily seen that the $\mathcal{J}(\tilde{U})$ is determined in A_N up to affine transformations. This proves Lemma 10.1.

Lemma 10.1 implies easily

THEOREM 10.2. *If an N-dimensional hypertangent bundle $\mathcal{J}(M)$ over a simply connected differentiable manifold M admits a locally flat hyperconnection ω, then there exists a submanifold $\tilde{M}$ immersed in the affine space A_N of N dimensions such that the pair $\{\mathcal{J}(M), \omega\}$ is equivalent to the pair $\{\mathcal{J}(\tilde{M}), \tilde{\omega}\}$, where $\mathcal{J}(\tilde{M})$ is the natural hypertangent bundle of $\tilde{M}$ and $\tilde{\omega}$ is the natural hyperconnection in $\mathcal{J}(\tilde{M})$. For the given $\mathcal{J}(M)$ and the locally flat hyperconnection ω, the natural hypertangent bundle $\mathcal{J}(\tilde{M})$ is determined up to affine transformations of A_N.*

The natural hyperconnection introduced in the natural hypertangent bundle of any submanifold immersed in an affine space is obviously locally flat. Then, we have

THEOREM 10.3. *When the base space M is simply connected, a necessary and sufficient condition for an N-dimensional hypertangent bundle $\mathcal{J}(M)$ over M to admit a locally flat hyperconnection is that $\mathcal{J}(M)$ be equivalent to the natural hypertangent bundle $\mathcal{J}(\tilde{M})$ of a submanifold $\tilde{M}$ immersed in the N-dimensional affine space.*

11. **Metric connection.**

Let $\mathcal{J}(M)$ be an N-dimensional hypertangent bundle over a differentiable manifold M. The structure group of $\mathcal{J}(M)$ being the linear group $L(N)$, as is well-known, the structure group is reducible to the orthogonal group $O(N)$ [7]. Therefore, there exists in $\mathcal{J}(M)$ a tensor field G of (0, 2)-type, which is positive definite and symmetric, *i.e.*, the tensor field G has components G_{ji} which form a positive definite matrix (G_{ji}) and

$$G_{ji} = G_{ij},$$

with respect to an N-frame $\{e_h\}$ defined in each coordinate neighborhood of M. We call such a tensor field G_{ji} a *metric tensor* in $\mathcal{J}(M)$. We suppose now that there is given a metric tensor G_{ji} in $\mathcal{J}(M)$.

If we introduce G^{ih} by

$$G_{ji}\, G^{ih} = \delta_j^h$$

then G^{ih} is a tensor field of (2, 0)-type and symmetric, *i.e.*,

$$G^{ih} = G^{hi},$$

If the metric tensor G_{ji} is parallel with respect to a hyperconnection $\Gamma_{c}{}^{h}{}_{i}$ in $\mathcal{J}(M)$, then the connection $\Gamma_{c}{}^{h}{}_{i}$ is called a *metric hyperconnection.* A hyperconnection is metric, if and only if

$$\nabla_b G_{ji} = \partial_b G_{ji} - \Gamma_{b}{}^{h}{}_{i}\, G_{jh} - \Gamma_{b}{}^{h}{}_{j}\, G_{hi} = 0,$$

or, (11.1)

$$\nabla_b G^{ih} = \partial_b G^{ih} + \Gamma_{b}{}^{i}{}_{j}\, G^{jh} + \Gamma_{b}{}^{h}{}_{j}\, G^{ij} = 0.$$

In the hypertangent bundle $\mathcal{J}(M)$ with the metric tensor G_{ji}, there exists uniquely a field of m-planes $\tilde{T}_P$ which are orthogonal to the tangent planes T_P at any point P of the base space M. We now employ this field of m-planes $\tilde{T}_P$ as the vertical vector bundle $\tilde{T}(M)$. Let $\{B_c\}$ be the natural frame defined in a coordinate neighborhood U of M. Then, there exist in U m vector fields spanning the vertical planes $\tilde{T}_P$ at each point P of U

and these fields form a vertical frame $\{C_x\}$ in U. Two planes T_P and $\bar{T}_P$ being orthogonal to each other, each B_c is orthogonal to each C_x, *i.e.*,

$$G_{ji}\ B_c^j\ C_x^i=0,$$

where B_c^j and C_x^i are the components of B_c and C_x respectively.

On putting

$$g_{cb}=G_{ji}\ B_c^j\ B_b^i,\quad g_{zy}=G_{ji}\ C_z^j\ C_y^i,$$

we have a metric tensor g_{cb} of the base space M and a metric tensor g_{zy} defined in the vertical vector bundle $\bar{T}(M)$ consisting of vertical planes $\bar{T}_P$. The metric tensors g_{cb} and g_{zy} are said to represent *metric induced* from G_{ji} respectively in the base space M and in the vertical vector bundle $\bar{T}(M)$. By means of (4.2), (11.2) and (11.3), we obtain

$$G_{ji}=B_j{}^c\,B_i{}^b\,g_{cb}+C_j{}^z\,C_i{}^y\,g_{zy}. \tag{11.4}$$

We introduce now two tensors g^{ba} and g^{yx} of corresponding types by

$$g_{cb}\,g^{ba}=\delta_c^a,\qquad g_{zy}\,g^{yx}=\delta_z^x.$$

If we put

$$\begin{aligned} B_{bi} &= G_{ih}\,B_b{}^h, & C_{yi} &= G_{ih}\,C_y{}^h,\\ B^{ah} &= g^{ab}\,B_b{}^h, & C^{xh} &= g^{xy}\,C_y{}^h,\\ B^a{}_i &= g^{ab}\,G_{ih}\,B_b{}^h, & C^x{}_i &= g^{xy}\,G_{ih}\,C_y{}^h, \end{aligned} \tag{11.5}$$

then $B^a{}_i$ and $C^x{}_i$ are the same things as those defined by (4.1) and we have

$$g^{ba}=B^b{}_i\,B^a{}_h\,G^{ih},\qquad g^{yx}=C^y{}_i\,C^x{}_h\,G^{ih} \tag{11.6}$$

$$G^{ih}=B_b{}^i\,B_a{}^h\,g^{ba}+C_y{}^i\,C_x{}^h\,g^{yx}. \tag{11.7}$$

Let $\Gamma_c{}^h{}_i$ be a metric hyperconnection in $\mathcal{J}(M)$. If we take covariant derivatives of both the sides in the two equations (11.3), we have respectively

$$\nabla_d\,g_{cb}=0,\qquad \nabla_d\,g_{zy}=0 \tag{11.8}$$

by means of (11.1), because we have from (6.8) and (6.9)

$$(\nabla_d B_c{}^j)\,B_b{}^i\,G_{ji}=0,\qquad (\nabla_d\,C_z{}^j)\,C_y{}^i\,G_{ji}=0$$

as a consequence of (11.2). Recalling (6.1) and (6.2), we shall notice now

$$\begin{aligned} \nabla_d\,g_{cb} &= \overset{\circ}{\nabla}_d\,g_{cb}\\ &= \partial_d\,g_{cb}-\overset{\circ}{\Gamma}_d{}^a{}_b\,g_{ca}-\overset{\circ}{\Gamma}_d{}^a{}_c\,g_{ab},\\ \nabla_d\,g_{zy} &= \overset{*}{\nabla}_d\,g_{zy}\\ &= \partial_d\,g_{zy}-\overset{*}{\Gamma}_c{}^x{}_y\,g_{zx}-\overset{*}{\Gamma}_d{}^x{}_z\,g_{xy}, \end{aligned}$$

where $\overset{\circ}{\Gamma}$ and $\overset{*}{\Gamma}$ are respectively the induced connection and the vertical connection induced from the given metric hyperconnection $\Gamma_c{}^h{}_i$. Then, (11.8) shows that *the two connections* $\overset{o}{\Gamma}$ *and* $\overset{*}{\Gamma}$ *are metric.*

Differentiating covariantly both sides in the identity

$$B_b{}^h C^x{}_h = 0,$$

which is equivalent to (11.2), we obtain

$$(\nabla_c B_b{}^h)\, C^x{}_h + B_b{}^h (\nabla_c C^x{}_h) = 0.$$

This equation is reduced to

$$(\nabla_c B_b{}^h)\, C^x{}_h + g^{xy} B_{bh}\ (\nabla_c C_y{}^h) = 0$$

by means of (11.5) and (11.8). Substituting (6.8) and (6.9) in the left-hand side and taking account of (11.5), we have

$$H_{cb}{}^x + g_{ba}\, g^{xy} L_{c\cdot y}^{\ a} = 0,$$

which implies

$$L_{c\cdot y}^{\ b} = - H_{c\cdot y}^{\ b}, \tag{11·9}$$

where we have put

$$H_{c\cdot y}^{\ b} = g^{ba} g_{xy} H_{ca}{}^x.$$

We shall now show some new identities satisfied by the curvature tensor of the metric hyperconnection $\Gamma_c{}^h{}_i$. We have from (11.1) the identity

$$\nabla_d \nabla_c G_{ji} - \nabla_c \nabla_d G_{ji} = 0,$$

which is reduced to

$$K_{dci}{}^h G_{jh} + K_{dcj}{}^h G_{hi} = 0, \tag{11.10}$$

because the left-hand sides of the above two equations are equal to each other. In a similar way, we obtain the identities

$$K_{dca}{}^e g_{be} + K_{dcb}{}^e g_{ea} = 0, \tag{11.11}$$

$$K_{dcx}{}^z g_{yz} + K_{dcy}{}^z g_{zx} = 0. \tag{11.12}$$

Then, if we put

$$K_{dcji} = K_{dcj}{}^h G_{hi}, \qquad K_{dcba} = K_{dcb}{}^e g_{ea},$$

$$K_{dcyx} = K_{dcy}{}^z g_{zx},$$

it follows respectively from (11.10), (11.11), (11.12)

$$K_{dcji} + K_{dcij} = 0,$$

$$K_{dcba} + K_{dcab} = 0,$$

$$K_{dcyx} + K_{dcxy} = 0,$$

which are new algebraic identities other than (6.3) containing the curvature tensors of the metric hyperconnection.

By virtue of (11.9), the identities (6.10) and (6.12) are reduced respectively to

$$
\begin{aligned}
K_{dcb}{}^a &= R_{dcb}{}^a + \left(H_d{}^a{}_y H_{cb}{}^y - H_c{}^a{}_y H_{db}{}^y \right), \\
K_{dcb}{}^x &= \nabla_d H_{cb}{}^x - \nabla_c H_{db}{}^x, \\
K_{dcy}{}^x &= R_{dcy}{}^x + \left(H_{db}{}^x H_c{}^b{}_y - H_{cb}{}^x H_d{}^b{}_y \right), \\
K_{dcy}{}^a &= \nabla_d H_c{}^a{}_y - \nabla_c H_d{}^a{}_y.
\end{aligned}
\qquad (11.13)
$$

We shall prove some theorems concerning a metric hyperconnection. If a metric hyperconnection has no torsion, its induced connection is symmetric in the base space by virtue of Theorem 5.1. Moreover, the induced connection $\overset{\circ}{\Gamma}_c{}^a{}_b$ is metric with respect to the metric g_{cb} induced in the base space from the given metric hyperconnection. Therefore, as is well-known, $\Gamma_c{}^a{}_b$ is necessarily equal to the Riemannian connection $\{_c{}^a{}_b\}$, which is the Christoffel's symbol

$$\left\{ {a \atop c\ b} \right\} = \frac{1}{2} g^{ad} \left(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{cb} \right)$$

constructed from g_{cb}. We have thus

Theorem 11.1. *The connection induced in the base space from a metric hyperconnection is the Riemannian connection determined by the metric induced in the base space, if the metric hyperconnection has no torsion.*

We shall now prove the following

Theorem 11.2. *In any hypertangent bundle with a metric tensor G_{ji} there exists always a metric hyperconnection having no torsion.*

Proof. Let $\tilde{\Gamma}_c{}^h{}_i$ be an arbitrary hyperconnection in a hypertangent bundle $\mathcal{J}(M)$ over a differentiable manifold M and suppose that there is given a metric tensor G_{ij} in $\mathcal{J}(M)$. Then, it is easily verified that the hyperconnection $\Lambda_c{}^h{}_i$ defined by

$$\Lambda_c{}^h{}_i = \tilde{\Gamma}_c{}^h{}_i + \frac{1}{2} \left(\tilde{\nabla}_c G_{ki} \right) G^{kh}$$

is metric, where $\tilde{\nabla}_c$ denotes the operation of covariant differentiation with respect to $\tilde{\Gamma}_c{}^h{}_i$.

Let $\overset{*}{\Lambda}_c{}^x{}_y$ be the vertical connection induced in the vertical vector bundle $\mathcal{T}(M)$ from the metric hyperconnection $\Lambda_c{}^h{}_i$. Let $\{_c{}^a{}_b\}$ be the Christoffel symbol determined by the metric g_{cb} induced in the base space from the given metric tensor G_{ji}. We shall define another hyperconnection $\Gamma_c{}^h{}_i$ by

$$\Gamma_{c\ i} = (-\partial_c B_b + \{_{c\ b}\} B_a{}^b) B_h{}_i + (-\partial_c C_y + \Lambda_c{}^x{}_y C_x{}^h) C^y{}_i, \qquad (11.14)$$

substituting

$$H_{cb}{}^{x}=0, \qquad L_{c\cdot y}{}^{a}=0,$$

$$\overset{\circ}{\Gamma}{}_{c}{}^{a}{}_{b}=\{{}_{c}{}^{a}{}_{b}\}, \qquad \overset{*}{\Gamma}{}_{c}{}^{x}{}_{y}=\overset{*}{\Lambda}{}_{c}{}^{x}{}_{y}$$

in the right-hand side of (4.18). Differentiating covariantly both the sides in (11.7) with respect to the new hyperconnection, $\Gamma_{c}{}^{h}{}_{i}$, we obtain

$$\begin{aligned}\nabla_{d}G^{ih}&=\nabla_{d}\,(B_{c}{}^{i}B_{b}{}^{h}g^{cb}+C_{x}{}^{i}C_{y}{}^{h}g^{xv})\\ &=B_{c}{}^{i}B_{b}{}^{h}\nabla_{d}\,g^{cb}+C_{x}{}^{i}C_{y}{}^{h}\nabla_{d}\,g^{xy},\end{aligned} \tag{11.15}$$

because we have

$$\nabla_{d}B_{c}{}^{h}=0, \qquad \nabla_{d}C_{x}{}^{h}=0,$$

by virtue of (6.8) and (6.9), taking account of

$$H_{cb}{}^{x}=0, \qquad L_{c}{}^{a}{}_{y}=0.$$

On the other hand, we obtain

$$\nabla_{d}\,g^{cb}=\overset{\circ}{\nabla}_{d}\,g^{cb}=0, \qquad \nabla_{d}\,g^{xv}=\overset{*}{\nabla}_{d}\,g^{xy}=0,$$

since the two connections $\left\{{}_{c}{}^{a}{}_{b}\right\}$ and $\overset{*}{\Lambda}{}_{c}{}^{x}{}_{y}$ are metric. Therefore, it follows from (11.15)

$$\nabla_{d}\,G^{ih}=0,$$

which means that the hyperconnection $\Gamma_{c}{}^{h}{}_{i}$ is metric.

If we substitute (11.14) in both the sides of the first equation of (5.8), we have

$$S_{cb}{}^{h}=0,$$

That is to say, the hyperconnection $\Gamma_{c}{}^{h}{}_{i}$ has no torsion. Summing up, we have proved that the hyperconnection $\Gamma_{c}{}^{h}{}_{i}$ is metric and has no torsion at the same time. Thus, Theorem 11.2 is proved.

Concerning metric hyperconnections which are locally flat, we can prove the following theorem in a quite similar way as in the proof of Theorem 10.2.

THEOREM 11.3. *If an N-dimensional hypertangent bundle* $\mathfrak{J}(M)$ *over a simply connected differentiable manifold M admits a metric tensor and a metric hyperconnection* ω *which is locally flat, there exists a submanifold* $\tilde{M}$ *immersed in the N-dimensional Euclidean space* E_N *such that the pair* $\{\mathfrak{J}(M),\omega\}$ *is equivalent to the pair* $\{\mathfrak{J}(\tilde{M}),\tilde{\omega}\}$, *where* $\mathfrak{J}(\tilde{M})$ *is the natural hypertangent bundle of* $\tilde{M}$ *and* $\tilde{\omega}$ *is the natural hyperconnection of* $\mathfrak{J}(\tilde{M})$. *For the given* $\mathfrak{J}(M)$ *and the locally flat metric hyperconnection* ω, *the natural hypertangent bundle* $\mathfrak{J}(\tilde{M})$ *is determined up to motions in* E_N.

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THE LAST GEOMETRIC THEOREM OF POINCARÉ

By

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INTRODUCTION. In 1912 Poincaré enunciated his last geometric theorem. Having only succeeded in treating a variety of special cases, he gave out the theorem for the consideration of other mathematicians. It was G. D. Birkhoff who furnished, in 1913, a proof of this theorem in a plane, two-dimensional case (a ring between two circles). In several of his subsequent papers Birkhoff generalized this theorem to a plane ring bounded by two arbitrary simply closed curves, one within the other, to an infinite plane ring, two plane rings and to a limiting case of a ring, a plane curve surrounding the origin "O" whose points are not advanced by the transformation. This theorem is the only one which was generalized by Birkhoff to $n=2m$ dimensions ; in this case the theorem refers to m-dimensional surfaces.

The present work is concerned with the proof of the Poincaré theorem referring to a finite ring, bounded by two curves, in m-dimensional space, $m>2$. In particular, the following theorem is proved:

Theorem

"Assume a doubly-connected hyper-ring, $R^{(m)}$, lying on a given open hyper-surface $\Sigma^{(m)}$, located in a sub-domain $D^{(m)} \epsilon B$, described by means of m-coordinates $(x^1, x^2, \ldots . x^m)$, $m>2$, $x^i \epsilon B$, B denoting a Banach space ; the hyper-ring $R^{(m)}$ is bounded by two closed hyper-curves $S_i^{(m)}$, $(i=1, 2)$ having no common points one with each other, *i.e.*, non-intersecting. Assume a not one-to-one transformation of coordinates from $(x^i, \ldots, x^m)$ to $(\eta_1, \ldots, \eta_{m-1})$, *i.e.*, from a m-dimensional sub-space, $D^{(m)}$, to a $(m-1)$-dimensional sub-space, $D^{(m-1)}$. Assume that the hyper-ring $R^{(m)}$ in $D^{(m)}$ on $\Sigma^{(m)}$ is associated with a doubly-connected hyper-ring $R^{(m-1)}$ in $D^{(m-1)}$ on an open hyper-surface $\Sigma^{(m-1)}$, bounded by two closed hyper-curves $S_i^{(m-1)}$, $(i=1, 2)$, having no common points one with each other (non-intersecting). Suppose both hyper-rings are subject to transformations $T^{(m)}$, $T^{(m-1)}$, respectively, in the sense of the last geometric theorem of Poincaré. Then, if there exist at least two points $P_i^{(m-1)}$, $(i=1, 2)$ of the ring $R^{(m-1)}$ invariant under $T^{(m-1)}$, there exist at least two points, $P_i^{(m)}$, $(i=1, 2)$, of the ring $R^{(m)}$, associated with $P_i^{(m-1)}$, $(i=1, 2)$, invariant under $T^{(m)}$."

For $m=2$, the proof of the last geometric theorem of Poincaré furnished by G. D. Birkhoff, is valid. The proof, given below, involves actually the application of the elements of the theory of invariant groups to the existence proofs of solutions of the systems of partial differential equations subject to transformations which reduce the number of independent variables, *i.e.*, which change the dimensionality of the subspaces from m to $(m-1)$.

1. THE LAST GEOMETRIC THEOREM OF POINCARÉ

In 1912 Poincaré enunciated a theorem of great value, in particular for the restricted problem of three bodies. Having only succeeded in treating a variety of special cases, he gave out the theorem for the consideration of other mathematicians. It was G. D. Birkhoff who furnished a proof of this theorem. The theorem may be stated in the following form :

THEOREM 1.1

"Given a ring $C<a\leqslant r\leqslant b$ in the (r, θ) plane (r, θ being polar coordinates), and a one-to-one, continuous, area-preserving transformation T of the ring into itself, which advances points on $r=a$ and regresses points on $r=b$. Then there will exist at least two points of the ring invariant under T."

We pass directly to the results obtained by Birkhoff.

2. BIRKHOFF'S PROOF

It was Birkhoff who actually in 1913 (see a collection of papers of Birkhoff on the theorem of Poincaré in the list of references) furnished a proof of Poincaré's theorem. The proof follows the pattern indicated below (see Birkhoff, 1927, pp. 166-169).

We take $x=\theta$, $y=r^2$ as the rectangular coordinates of a point in the (x, y)-plane. The ring transforms into a strip $a^2\leqslant y\leqslant b^2$. The transformation T of this strip advances points on the boundary $y=a^2$ to the right, and moves points on $y=b^2$ to the left. The transformation T is an area-preserving transformation ($2rdrd\theta=dxdy$) and displaces any two points possessing the same ordinate and whose abscissas differ by a multiple of 2π in the same manner. Let us assume a second transformation T_ϵ which effects a translation of the (x, y)—plane in the direction of the y-axis through a distance $\epsilon>0$. The transformation TT_ϵ is an area-preserving transformation which shifts the given strip into the strip $a^2+\epsilon\leqslant y\leqslant b^2+\epsilon$. Suppose that there exists no invariant point of T. Then there exists a positive quantity "d" such that all points are displaced at least a distance "d" by the transformation T. Birkhoff chooses $\epsilon<d$. Consider now a narrow strip $a^2\leqslant y\leqslant a^2+\epsilon$. By the transformation TT_ϵ the lower edge of this strip is carried into the upper edge and the strip is carried into a second strip lying wholly above the first one save along the common edge. By a repetition of TT_ϵ the second strip goes into the third, etc. By a continuation of this process one obtains a series of strips (consecutive strata) which are unaltered by a shift of 2π to the right, since T and T_ϵ are single-valued over the ring. The images of these strata on the ring represent a set of closed strata lying one within the other, all having equal area since TT_ϵ is an area-preserving transformation in both planes, (r, θ) and (x, y). This implies that some one of the strata on the infinite strip, say the k-th, will overlap the upper edge $y=b^2$. Let P be the point of $y=a^2$ which is being moved in the (x, y)-plane by means of the strata, described above, into a point Q on the upper edge of the k-th stratum for which y is a maximum. Obviously, the location of Q is obtained after a k-fold repetition of TT_ϵ. The points P', P'', $\ldots . P^{(k)}\equiv Q$ represent the successive images of P under

the iteration of TT_ϵ. The line PP' lies in the first stratum, the successive images of this line, $P'P''$, etc. lie in the successive strata, and have no points in common except that successive arcs have an end point in common. Thus, we get a single arc PQ, made up of all these lines, which is without double points.

Consider now a vector LL' drawn from a point L to its image L' under TT_ϵ, of which the initial point moves from P to $P^{(k-1)}$ along the line PQ. The angle which this vector makes with the positive direction of the x-axis at the outset is a positive acute angle (P' is to the right of and above P). When L has varied to its final position $P^{(k-1)}$, this angle lies in the second or third quadrant, since $P^{(k)}$ lies to the left of $P^{(k-1)}$ by the hypothesis of the theorem. Thus, LL' has rotated through the least positive angle from the first direction to the second. Now let ϵ approach zero. As ϵ becomes smaller the vector LL' continues to have a definite direction, since no invariant points under TT_ϵ are present. By a limiting process we infer that, for the transformation T, the angular variation of LL' is through the least positive angle consistent with its initial and final directions. For L on $y=a^2$ the direction of LL' is that of the positive x-axis, while on $y=b^2$ the direction is that of the negative x-axis.

Consider the inverse transformation T^{-1} which is of the same type as T but it moves points on $y=a^2$ to the left and points on $y=b^2$ to the right. By the argument analogous to that given above we conclude that if a vector $LL^{(-1)}$ with the end-point $L^{(-1)}=T^{(-1)}L$ has its initial point L varied from a point of $y=a^2$ to a point $y=b^2$, the total angular variation will be the least negative angle consistent with its initial and final positions. But the total rotation of $LL^{(-1)}$ is the same as that of the oppositely directed vector $L^{(-1)}L$ which joins a point $L^{(-1)}$ of $y=a^2$ to its image L under T. On the other hand, by our earlier results, the total angular variation of $L^{(-1)}L$ must also be the least positive angle consistent with the two positions. Hence, we have been led to a contradiction, so that there must exist at least one invariant point. Poincaré had shown that there are at least two invariant points. Let the point P describe the fundamental rectangle, $0<x<2\pi$, $a^2<y<b^2$, in the (x, y)-plane, in a positive sense; the total rotation of the vector LL', is zero over this circuit, since there is no rotation along $y=a^2$ or $y=b^2$, and the rotations along $x=0$ and $x=2\pi$ are the negatives of one another. But around a simple invariant point the rotation is $\pm 2\pi$. Hence, there will either be at least two simple invariant points with rotations $+2\pi$ and -2π, or there will be at least one multiple invariant point. Both Poincaré and Birkhoff demonstrate that there will always be at least two geometrically distinct invariant points.

3. GENERALIZATION BY BIRKHOFF

Birkhoff proposes a generalization of the original theorem of Poincaré in the following sense [see his paper in 1913, p. 22] : The circles of the radii $r=a$, $r=b$, may be allowed to be any simply closed curves, one within the other, bounding a ring R. We may again state a similar theorem, for we can make a preliminary transformation, for instance a conformal one, to take these curves into concentric curves, when of course the integral invariant merely changes form. Birkhoff mentions that it may be necessary to consider with care the nature of the integral invariant near the boundaries after the transformation of coordinates. In 1917 Birkhoff modified the theorem of Poincaré in a more general sense. The modification of Birkhoff is the following (Trans. A.M.S., 1917, p. 294) :

THEOREM 3.1

"Given an infinite ring $0<a<r$ in the (r, θ)-plane, and a one-to-one continuous, area-preserving transformation T of the ring into itself, which advances points on $r=a$ and regresses all points $a<R<r$ by at least an angle $\theta_1>0$. Then there will exist at least two points of the ring $a<r<R$ invariant under T."

The proof of this theorem is exactly the same as the one given above.

In 1925 Birkhoff provided a further generalization of Poincaré's theorem. Let us consider in the (r, θ)-plane a doubly* connected ring R, bounded by the circle C of the radius "a" and a closed curve Γ encircling C, as well as a second like ring R_1 bounded by the same circle C and a like encircling curve Γ_1. The two rings R and R_1 are taken to be related so that a one-to-one direct, continuous point transformation T carries R into R_1. We write :

$$C_1 = T(C)\ ;\ \Gamma_1=T(\Gamma)\ ;\ R_1 = T(R)\ ; \tag{3.1}$$

$$C = T^{(-1)}(C_1)\ ;\ \Gamma = T^{(-1)}(\Gamma_1)\ ;\ R = T^{(-1)}(R_1). \tag{3.2}$$

The Birkhoff extension of Poincaré's last geometric theorem is as follows :

THEOREM 3.2

"If Γ and Γ_1 are met only once by any radial line θ = constant, and if T carries points on C and Γ in opposite angular directions with respect to θ to their new positions on C and Γ_1 respectively, then either (a) there are two distinct invariant points P of R and R_1 under T, or (b) there is a ring in R (or R_1) abutting upon C which is carried into part of itself by T (or $T^{(-1)}$)."

In the form enunciated by Poincaré the boundaries Γ and Γ_1 coincide, while the alternative (b) is excluded by means of the hypothesis that an area integral is invariant under T. It is worthy of note that the extended theorem does not involve the hypothesis of an invariant area integral, and so falls essentially in the domain of analysis situs.

The proof of the theorem, given above, can be found in Birkhoff's paper (1925).

In 1931 Birkhoff furnished another form of a generalization of Poincaré's theorem.

Suppose T is a continuous, one-to-one and direct transformation of a region R into a region $T(R)$, R being simply connected and surrounding an invariant point 'O'. Suppose moreover T is a conservative transformation, *i.e.*, preserves the areas. Then the theorem of Poincaré may be expressed in the following form :

---

* A closed curve is defined as the common boundary of a finite, simply connected, open continuum and the complementary open outer continuum. A ring is the region bounded by two closed curves, one within the other. If these curves do not touch, the ring is a doubly connected open continuum.

THEOREM 3.3

"If a transformation T rotates all the points of the bounding curve of R in one and the same direction around O, and all the points sufficiently close to O in the opposite direction, then there exist at least two other invariant points."

Birkhoff demonstrates that obviously this theorem is closely tied with the following theorem :

THEOREM 3.4

"If the transformation T does not advance around O none of the points of a closed curve, C, surrounding O, and cut only once by each radial line originated in O, then there exist at least two invariant points on such a curve."

The demonstration of this theorem is immediate. In reality, the image $T(C)$ cannot contain C in its interior, as well as it cannot be contained inside C, due to the invariance of areas. This implies that $T(C)$ must cut the curve C at least in two points, and the points of intersection are necessarily the invariant points.

Birkhoff indicates how this theorem can be generalized to $n=2m$ dimensions.

4. INVARIANT GROUPS AND REDUCTION OF INDEPENDENT VARIABLES

The author of the present work was inspired by some results obtained in the field of differential equations to apply the concept of the reduction of the number of independent variables to the present problem. Dickson in 1924 (see references) showed how some differential equations can be integrated with the aid of group theory establishing a link between the techniques of algebra and those of differential equations. In 1949 Garrett Birkhoff suggested that the reduction of independent variables in systems of partial differential equations could be attacked by algebraic methods. The method of reducing the number of independent variables by means of a transformation from one space into another does not appear amenable to the standard methods of the classical theory of functions since the Jacobian of the transformation vanishes. However, algebraic methods do not break down under the above conditions.

Michal (see references) undertook the following task : "Given a differential system in the space A of n-dimensions, transform this system into the space B of $(n-r)$-dimensions $(1<r)$ such that a solution of the system in B determines a solution of it in A. The transformations involved are not to be one-to-one." The resulting theorems involve operations in the Banach space and some elements of the theory of continuous transformation group.

Morgan (see references) applied the above method to two practical cases where the transformation was from n to $(n-1)$ dimensions. In one of his papers (v. Krzywoblocki and Roth, see references) the author of the present work applies the general theory to some specific problems. In particular, he deals with the transformation of systems of partial differential equations of n-th order into systems of ordinary differential equations through the use of one-parameter groups. Each appropriate one-parameter group reduces the number of independent variables in the system in question by one by transforming the variables of the group into the invariants of the group. A method is developed for obtaining the proper groups for a given system of partial differential equations.

The general fundamentals of this technique are applied below to the problem which forms the main subject treated in the present paper. Before we shall provide a proof of the generalized Poincaré last geometic theorem, we have to put down a few mathematical notions necessary to proceed with the derivation of the proof. Suppose we deal with a transformation of coordinates of the form :

$$T : \bar{x}^i = f^i(x^1, .., x^n \; ; \; a^1, .., a^j), \; (i=1, \; .., \; n), \tag{4.1}$$

where $x^i \epsilon B_i$, $\bar{x}^i \epsilon \bar{B}_i$, and the B_i's and the $\bar{B}_i$'s are Banach spaces. The numbers a^i's are parameters. Equation (4.1) may then be thought of as representing a transformation of the product space P $(B_1 \times B_2 \times \ldots . \times B_n)$ into the product space $\bar{P}$ $(\bar{B}_1 \times \bar{B}_2 \times \ldots \ldots \times \bar{B}_n)$. If a transformation T transforms a neighborhood of P into a neighborhood of $\bar{P}$ then the transformation is said to be a continuous one. A transformation T, Equation (4.1), denotes a group if and only if it possesses an inverse, contains an identity and satisfies the group properties, *i.e.*,

(1) there exists an inverse transformation :

(2) there exists identity element ;

(3) group property, *i.e.*, the product transformation is also a transformation of T.

Consider an r-parameter continuous group of transformations :

$$\Gamma_r : \bar{x}^i = f^i(x^1, \; .., \; x^m \; ; \; a^1, .., a^r) = f^i(x \; ; \; a) \; ; \tag{4.2}$$

$$\bar{y}^j = f^j(y \; ; \; a), \; (i=1, \; .., m; \; j=1, .. , n) \; ; \; m \geqslant 2. \tag{4.3}$$

Obviously, the group Γ_r possesses a sub-group, $S\Gamma_r : x^i \longleftrightarrow \bar{x}^i$. This is exactly the transformation group in question $(\bar{x}^i = f^i(x \; ; a))$ which is independent of y. If a numerically valued (scalar) function F,

$$F = F(x^1, \; .., \; x^m \; ; \; y^1, \; .., \; y^n), \tag{4.4}$$

is unaltered by the transformation Γ_r, *i.e.*, if $F(x^1, .., x^m, y^1, .., y^n) = F(\bar{x}^1, .., \bar{x}^m, \bar{y}^i, .., \bar{y}^n)$, under Γ_r, then the function $F(x, y)$ is called an absolute invariant of Γ_r. (4.5)

Let us consider a set of $m+n$ functions, $F_1(x, y)$, .., $F_{m+n}(x, y)$; the functions F_k are said to be independent if and only if the determinant :

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial x^i} & \dfrac{\partial F_{m+n}}{\partial x^i} \\ \dfrac{\partial F_1}{\partial y^j} & \dfrac{\partial F_{m+n}}{\partial y^j} \end{vmatrix} \neq 0, \; (i=1, \; .., m \; ; \; j=1, \; .., \; n). \tag{4.6}$$

There are various terms used for defining the concept of independent functions, like the term "functionally independent" [see Morgan's paper, References]. To the sub-group $S\Gamma_r$ [*Eq.* (4.3)], (in a Banach space), the following theorem applies :

THEOREM 4.1

The group $S\Gamma_r$ has $(m-p)$ functionally independent, differentiable (and therefore continuous) absolute invariants $(\eta_1, .., \eta_{m-p})$, where $p(p<r)$ is the rank of the matrix :

$$\left|\begin{matrix}\dfrac{\partial x^i}{\partial a^1}\\ \dfrac{\partial x^i}{\partial a^r}\end{matrix}\right|, \quad (i=1, .., m) \tag{4.7}$$

The proof of this theorem is in Eisenhart [Ref., pp. 20-21, 61-62]. The successive reduction through one-or multi-parameter groups of the number of independent variables particularly in a system of partial differential equations was elaborated by V. Krzywoblocki and Roth [see References]. Here we can propose that the $(m-1)$-tuply invariant transformations and solutions of partial differential equations may be reduced to solutions of ordinary differential equations. The author would like to point out that the technique, presented above, is actually inseparably associated with the fundamental notions of the synthetic (projective) geometry. In reality, a n-dimensional surface can be projected (parallelly or centrally, *i.e.*, by the use of the perspective geometry) from the n-dimensional space into $(n-1)$-dimensional space, thus furnishing an association between surfaces of different dimensions (n to $n-1$). As an example, a projection of a sphere on a plane furnishes a circle. Synthetic geometry provides an association between geometric elements of different dimensions (a pencil of lines or planes and a set of points on a straight line, etc.). Baer [see references] had actually shown the existing relations between linear algebra and projective geometry. This item generalizes in a still broader sense the concept of the reduction of independent variables in differential systems. A ring located on a sphere corresponds to a ring located in the circle. A transformation of a ring on the sphere into itself in the sense of Poincaré's theorem corresponds to the transformation of the ring in the circle into itself in the sense of Poincaré. It seems that there is the entire class of possible solutions of this problem as there is the entire class of possible projections of sphere onto a plane. Some of those projections cannot be accepted. A circle located on a sphere may be projected as a finite segment of a straight line. Such an association of the (m)-and $(m-1)$-spaces has to be rejected.

5. N-DIMENSIONAL HYPER-RING

The use of topological methods in proving existence theorems like (as an illustrative example) the "fixed point method," related to ideas of Poincaré, was already envisioned by G. D. Birkhoff and Kellogg, [see references], and developed into a powerful tool by Schauder and Leray [see references]. The Schauder fixed point theorem is an extension to infinite dimensional spaces of a celebrated theorem due to Brouwer.

The proof of the Brouwer fixed point theorem can be found in most texts on topology. The proof of the Schauder fixed point theorem is given, between other references, in [Courant, p. 403, see references]. Both the theorems were announced years after the last geometric theorem of Poincaré, which refers to two fixed points. Below, we attempt to prove

Poincaré's theorem—at least in some particular cases—in n-dimensional space. The considerations, given below, represent formally some sort of a continuation of the discussion contained in previous chapters, and consequently refer to a Banach space, B. We propose to use the following symbols : S for a closed curve, Σ for an open surface. Assume that any system in B is described by means of $(x^1, x^2, .., x^m)$, $m>2$, independent variables ; the function $y(x^1, x^2, .., x^m ; a) \equiv \Sigma^{(m)}$ denotes an open hypersurface depending upon x's and on a parameter a. Let us also assume that on $\Sigma^{(m)}$ there is located a doubly-connected hyper-ring $R^{(m)}$, bounded by two closed hyper-curves, $S_1^{(m)}$ and $S_2^{(m)}$, having no common points one with each other, *i.e.*, non-intersecting.

Assume a one-parameter group :

$$G_1 : \bar{x}^i = f^i(x ; a) ; \bar{\Sigma}^{(m)} = f^y(y ; a) ; \bar{S}_j^{(m)} = f^s(\bar{S}_j'^{(m)}; a), \tag{5.1}$$
$$j = 1, 2, \ x = x^1, .., x^m,$$
$$y \equiv y_\delta \equiv \Sigma^{(m)}, \ \delta = 1, 2, 3, \ ..,$$

which possesses a sub-group, S_{G_1}, $x^i \rightarrow \bar{x}^i$. This sub-group is assumed to possess $(m-1)$ functionally independent, Fréchet differentiable, absolute invariants,

$$\eta = \eta_1(x^1, .., x^m), \ .., \ \eta_{m-1}(x^1, .., x^m). \tag{5.2}$$

Suppose that we now consider the $y_\delta \equiv \Sigma^{(m)}$, $S_j^{(m)}$, $(j=1, 2)$, and $\bar{y}_\delta \equiv \bar{\Sigma}^{(m)}$, $\bar{S}_j^{(m)}$, $(j=1, 2)$, $\delta = 1, 2, 3$, to be implicitly defined as functions of the x^i and $\bar{x}^i$, respectively, by the equations :

$$z_\delta(x^1, .., x^m) = g_\delta(y_1, y_2, y_3, x^1, .., x^m) ; \tag{5.3}$$
$$\bar{z}_\delta(x^1, .., x^m) = g_\delta(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{x}^1, .., \bar{x}^m), \tag{5.4}$$

where the g_δ are the absolute invariants of the group G_1, *i.e.*, the group G_1 possesses the $(m+3-1)$ absolute invariants :

$$\eta_1(x^1, .., x^m), .., \eta_{m-1}(x^1, .., x^m),$$
$$g_1(y_1, y_2, y_3, x^1, .., x^m), \ .., \ g_3(y_1, y_2, y_3, x^1, .., x^m). \tag{5.5}$$

We then apply the theorem proved by Morgan [see references] :

THEOREM 5.1

A necessary and sufficient condition of the y_δ, implicitly defined as functions of $x^1, .., x^m$ by the relations (5.3), to be exactly the same functions of $x^1, .., x^m$, as the $\bar{y}_\delta$, implicitly defined as functions $\bar{x}^1, .., \bar{x}^m$ by the relations (5.4), are of the $\bar{x}^1, .., \bar{x}^m$ is that :

$$z_\delta(x^1, .., x^m) = \bar{z}_\delta(\bar{x}^1, .., \bar{x}^m) = z_\delta(\bar{x}^1, .., \bar{x}^m). \tag{5.6}$$

The condition (5.6) can be replaced by :

$$z_\delta(x^1, .., x^m) = F_\delta(\eta_1, .., \eta_{m-1}), \ \delta = 1, 2, 3, \tag{5.7}$$

where the η's are the set of absolute invariants of the sub-group $x^i \rightarrow \bar{x}^i$ of the group G_1.

The proof of this theorem is in Morgan's work [references, p. 251].

It is easy to notice that the transformation of coordinates from the m-dimensional space to the $(m-1)$-dimensional space is not one-to-one. This implies that we shall be able to find more than one set of the coordinate transformation functions $x^i \rightarrow \bar{x}^i$. With this in mind we assume that the hyper-ring $R^{(m-1)}$ in the $(m-1)$-dimensional space associated through Eq., (5.7) with the hyper-ring $R^{(m)}$ in the m-dimensional space is bounded by two closed hyper-curves, $S_1^{(m-1)}$ and $S_2^{(m-1)}$ having no common points one with each other, and that the surface $\Sigma^{(m-1)}$ is an open one. Assume that the hyper-ring $R^{(m-1)}$ is transformed in the sense expressed in the last geometric theorem of Poincaré*. Moreover, let us assume that under these conditions we had shown the existence of at least two invariant points of $R^{(m-1)}$. Then we have a theorem :

THEOREM 5.2

Under the conditions explained above there exist at least two invariant points of $R^{(m)}$.

The proof of this theorem is a straightforward one. Let us denote the coordinates of these two points in the $(m-1)$ space by the symbols :

$$\eta\,(P_i^{(m-1)}) = \eta_1\,(P_i^{(m-1)}),\ ..,\ \eta_{m-1}\,(P_i^{(m-1)}),\ i=1,\ 2. \tag{5.8}$$

Since they are known and given, we obtain with the use of *Eq.*, (5.2) $(m-1)$ equations for the m-coordinates of the two associated points in the m-dimensional space, *i.e.*,

$$x(P_i^{(m)}) = x^1(P_i^{(m)}),\ ..,\ x^m(P_i^{(m)}),\ i=1,\ 2. \tag{5.9}$$

The last required equation is furnished by the condition that the points $P_i^{(m)}$ must be located on $\Sigma^{(m)}$. Thus :

$$\Sigma^{(m)} = y(x^1\,(P_i^{(m)}),\ ..,\ x^m\,(P_i^{(m)})\ ;\ a),\ i=1,\ 2. \tag{5.10}$$

Hence, the last geometric theorem of Poincaré in the extended sense is proved, for any $m>2$; for $m=2$ the proof was provided by George Birkhoff.

This proves the theorem enunciated in the introduction of the present work.

There remains the practical problem of finding the invariant groups in particular cases. This is not an easy problem and it will not be discussed here. In his and Roth's work [see references] the author provides a thorough presentation of numerous particular cases of finding the invariant groups in various fields of mathematical physics.

The proof, presented above, refers to a hyper-ring situated on an open hyper-surface. There remains the problem of proving the last geometric theorem of Poincaré for a hyper-sphere, *i.e.*, hyper-ring contained between two closed, non-intersecting hyper-surfaces. This problem will be treated in the future in a separate paper.

The theorem, presented in the introduction and proved in Section 5 does refer to hyper-rings located on open hyper-surfaces. But it seems that a certain caution should be applied in using it. It certainly is valid if the transformation between $R^{(m)}$ and $R^{(m-1)}$ is one-to-one.

Example : Assume a ring located on a sphere projected on the horizontal plane. The ring $R^{(m)}$ is located above the equator of the sphere, possesses two horizontal circles (cross-sections of the sphere) as its boundary curves ; the centers of these two circles are located on the vertical axis of the sphere. But if one of the two boundary curves of $R^{(m)}$ is located below the equator, then the mapping between $R^{(m)}$ (spherical ring) and $R^{(m-1)}$ (circular plane ring) is not one-to-one. The circular ring is a two-sheet surface (ring) analogous to a Riemann surface in the theory of complex variable functions. It possesses a fold along the circumference of the largest sphere corresponding to the equator of the sphere. Such cases seem to require an additional attention.

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* Obviously, this implies that the hyper-ring $R^{(m)}$ transforms into itself in the same sense.

By a proper cut of the sphere in the first case, cited above, (ring located above the equator) and by means of the rejection of the lower part of the sphere one can establish a one-to-one correspondence between the sphere and its projection, *i.e.*, a circle.

6. BIRKHOFF'S THEOREMS

It is obvious that all the other theorems, but one, proved by Birkhoff, can be extended to m-dimensional space. Thus, the theorems referring to the following cases are valid :

(i) A ring bounded by any simply closed hyper-curves in m-dimensional space ;
(ii) Theorem 3.1a. — Corresponding to Theorem 3.1 in m-dimensions ;
(iii) Theorem 3.2a. — The same ;
(iv) Theorem 3.3a. — The same ;
(v) Theorem 3.4a. — The same.

But, as yet, the present technique does not work in the case of the last Birkhoff's theorem explained below theorem 3.4, *i.e.*, a generalization to $n = 2m$ dimensions.

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ON NEW THEORIES OF SPACE IN THE UNIFIED FIELD THEORY (*)

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1. Introduction

In his last theory of unified field, Einstein (Einstein, 1956) proposed the following field equations for the four dimensional space :

$$g_{\lambda\mu.\nu} - g_{\alpha\mu}\Gamma^{\alpha}_{\lambda\nu} - g_{\lambda\alpha}\Gamma^{\alpha}_{\nu\mu} = 0, \tag{1}$$

$$\Gamma^{\alpha}_{[\lambda\alpha]} = 0, \qquad R_{(\lambda\mu)} = 0, \quad R_{[\lambda\mu]} = X_{[\lambda,\mu]}, \tag{2}$$

where $g_{\lambda\mu}$ is a non-singular asymmetric tensor, $\Gamma^{\rho}_{\lambda\mu}$ is an asymmetric connection, $R_{\lambda\mu} = R^{\alpha}_{\lambda\mu\alpha}$, where $R^{\rho}_{\lambda\mu\nu}$ is the curvature tensor formed with $\Gamma^{\rho}_{\lambda\mu}$ and X_{λ} is any covariant vector. Further, comma followed by indices denotes the ordinary derivative and the two brackets, ordinary and square, used in (2) signify respectively the symmetric and the skew-symmetric parts of the entities to which they are attached. Hlavatý (Hlavatý, 1957), in his book on geometry of the unified field theory, has given a detailed exposition and implications of the above field equations.

It has been our object to study some general theories of space of n dimensions that arise from consideration of the above field equations and to present some equations and theorems obtained by Hlavatý from a more general point of view. We introduced in a previous paper (Sen, 1964) a concept of generalised covariant differentiation by means of which we reformulated some equations given by Hlavatý in connection with the equations (1). In the present paper a concept of generalised curvature tensors based on generalised cavariant differentiation has been introduced by means of which some equations given by Hlavatý in connection with both equations (1) and (2) have been reformulated. Finally, by means of these concepts, a generalised form of the field equations (1), (2) has been given.

It seems desirable to give here an outline of the generalised covariant differentiation mentioned and referred to above as follows :

In an n-dimensional space let there exist several affine connections denoted by ${}^{a}\Gamma^{\rho}_{\lambda\mu}$, ${}^{b}\Gamma^{\rho}_{\lambda\mu}$, ${}^{c}\Gamma^{\rho}_{\lambda\mu}$,......, where $a, b, c,$...... are symbols distinguishing the different connec-

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tions. To form a generalised covariant derivative of a tensor, say $T^{\rho\cdot\,\cdot\cdot\,\cdot}_{\lambda\mu}$, with respect to one or more of these affine connections, we take the scalar

$$T^{\rho\cdot\,\cdot\,\cdot\cdot}_{\lambda\mu\,..}\,u_{\rho}\,v^{\lambda}w^{\mu}\ldots\ldots,$$

where u_ρ, v^λ, w^μ,.........are arbitrary vectors, and find the increment of the scalar when the vectors are given the desired parallel displacements in any direction dx^ν. Then the expression $U^{\rho\cdot\cdots\cdot}_{\lambda\mu\ldots\nu}$ appearing as coefficient of the increment, say

$$\left(U^{\rho\cdot\,\cdots}_{\lambda\mu\quad\nu}\right)u_\rho v^\lambda w^\mu\ldots\ldots dx^\nu,$$

is the required generalised covariant derivative. More precisely if, for example, the vector u_ρ, v^λ, w^μ,.........are given the parallel transports corresponding to ${}^{a}\Gamma^{\rho}_{\lambda\mu}$, ${}^{b}\Gamma^{\rho}_{\lambda\mu}$, ${}^{c}\Gamma^{\rho}_{\lambda\mu}$, ... respectively, then it is seen

$$U^{\rho\ldots}_{\lambda\mu\ldots\nu}=T^{\rho\cdot\,\cdot}_{\lambda\mu\cdot\ ,\nu}+T^{\alpha\cdot\,\cdot}_{\lambda\mu\cdot\cdot}\,{}^{a}\Gamma^{\rho}_{\alpha\nu}+\ldots-T^{\rho\cdot}_{\alpha\mu\ \cdot\cdot}\,{}^{b}\Gamma^{\alpha}_{\lambda\nu}-T^{\rho\cdot}_{\lambda\alpha\ldots}\,{}^{c}\Gamma^{\alpha}_{\mu\nu}-\ldots.$$

We shall always denote the ordinary derivative by a comma and the covariant derivative by a semi-colon. In a generalised covariant derivative of a tensor, symbols specifying the affine connections will be placed below the indices of the tensor to indicate the corresponding parallel transports which have been carried out. Coordinate indices will be denoted by Greek letters and their symbols by Roman letters. Thus the above covariant derivative will be denoted by

$$U^{\rho\cdot\cdot}_{\lambda\mu\cdot\cdot\,\nu}=\underset{bc\ \ldots}{\overset{\rho\ \ldots}{T^{a\ \ldots}_{\lambda\mu\cdot\cdot\ ;\nu}}}$$

Generalised covariant derivative has the following properties :

(1) Generalised covariant derivative of a tensor is a tensor.

(2) Generalised covariant derivative of the sum, difference, outer and inner products of tensors obeys the same laws as ordinary derivatives.

(3) Generalised covariant derivative of a scalar is the ordinary derivative as long as every pair of indices of contraction, if any, have the same symbol.

Repeated generalised covariant derivative of a tensor will be denoted by placing the desired symbols below the given ones. Thus a generalised covariant derivative of the above tensor will be denoted by

$$V^{\rho\,\cdot}_{\lambda\mu\quad\nu w}=\underset{q\,r..\,s}{\overset{\rho\ \ldots}{U^{p\,\cdot\cdot}_{\lambda\mu\quad\nu;\,w}}}=\underset{\substack{bc\ \ldots\ s\\ qr\ \ \ldots}}{\overset{\substack{\rho\ \ldots\\ a\ \ldots}}{T^{p\ \ldots}_{\lambda\mu\ \ldots;\,\nu w}}}$$

2. Certain geometrical entities

By forming repeated generalised covariant derivative of a covariant vector ν_λ in the anner stated in the last section, we find

$$\nu_{\underset{q}{\underset{p}{\lambda}}\,;\,\underset{r}{\mu}\nu}-\nu_{\underset{q'}{\underset{p'}{\lambda}}\,;\,\nu\underset{s}{\mu}}=\nu_a\left({}^{p'}\Gamma^a_{\lambda\nu,\mu}-{}^{p}\Gamma^a_{\lambda\mu,\nu}+{}^{p}\Gamma^a_{\beta\mu}\,{}^{q}\Gamma^\beta_{\lambda\nu}-{}^{p'}\Gamma^a_{\beta\nu}\,{}^{q'}\Gamma^\beta_{\lambda\mu}\right)$$

$$+\nu_{a,\mu}\left({}^{p'}\Gamma^a_{\lambda\nu}-{}^{q}\Gamma^a_{\lambda\nu}\right)+\nu_{a,\nu}\left({}^{q'}\Gamma^a_{\lambda\mu}-{}^{p}\Gamma^a_{\lambda\mu}\right)-\nu_{\underset{p}{\lambda}\,,\,a}\,{}^{r}\Gamma^a_{\mu\nu}+\nu_{\underset{p'}{\lambda}\,;\,a}\,{}^{s}\Gamma^a_{\nu\mu}\,. \tag{2.1}$$

The expression appearing as coefficient of ν_a in the first line of (2.1), namely

$$C^a_{\lambda\mu\nu}(p,q,p',q')={}^{p'}\Gamma^a_{\lambda\nu,\mu}-{}^{p}\Gamma^a_{\lambda\mu,}+{}^{p}\Gamma^a_{\beta\mu}\,{}^{q}\Gamma^\beta_{\lambda\nu}-{}^{p'}\Gamma^a_{\beta\nu}\,{}^{q'}\Gamma^\beta_{\lambda\mu}, \tag{2.2}$$

may be seen to come out from the following geometical consideration :

Consider an infinitesimal parallelogram $PQRS$ whose adjacent sides PQ and PS, regarded as vectors having P as the initial point, are the elementary vectors dx^i and δx^i respectively. Let a contravariant vector u^a at P be given the parallel displacement corresponding to ${}^{p}\Gamma^\rho_{\lambda\mu}$ along PQ and then parallel displacement corresponding to ${}^{q}\Gamma^\rho_{\lambda\mu}$ along QR. Then the change in u^a on passing from P to R along the path PQR is given by

$$\delta du^a=-\left({}^{p}\Gamma^a_{\lambda\mu,\nu}-{}^{p}\Gamma^a_{\beta\mu}\;{}^{q}\Gamma^\beta_{\lambda\nu}\right)u^\lambda\,dx^\mu\,\delta x^\nu-{}^{p}\Gamma^a_{\lambda\mu}\,u^\lambda\delta dx^\mu.$$

Similarly for the change in the vector when it is given parallel displacements corresponding to ${}^{p'}\Gamma^\rho_{\lambda\mu}$, ${}^{q'}\Gamma^\rho_{\lambda\mu}$ successively along PS, SR respectively. Therefore when the vector is transported round the parallelogram, the change in the vector is given by

$$(\delta d-d\delta)u^a=C^a_{\lambda\mu\nu}(p,q,p',q')u^\lambda dx^\mu\,\delta x^\nu-\left({}^{p}\Gamma^a_{\lambda\mu}\;\delta dx^\mu-{}^{p'}\Gamma^a_{\lambda\mu}\;d\delta x^\mu\right)u^\lambda, \tag{2.3}$$

where $C^a_{\lambda\mu\nu}(p,q,p',q')$ is defined by (2.2). If, however, $p=p'$, the second term on the right hand side of (2.3) may be omitted by neglecting the very small quantity $(\delta d-d\delta)\,x^\mu$. This is the required geometrical significance.

Suppose now each of the symbols p, q, p', q' is either a or b which are supposed to be two distinct symbols. The different cases that may arise are the following :

$$\begin{aligned}(p,\,q,\,p',\,q')\\ =&\text{(i) }(a,b,a,b),\ \text{(ii) }(a,b,b,a,),\ \text{(iii) }(a,b,a,a,),\ \text{(iv) }(a,b,b,b),\\ &\text{(v) }(a,a,a,b),\ \text{(vi) }(a,a,b,a),\ \text{(vii) }(a,a,b,b),\ \text{(viii) }(a,a,a,a)\end{aligned}\Bigg\} \tag{2 4}$$

and 8 other cases that are obtained from (2.4) by interchanging a, b. Let us denote the 8 expressions (2.2) corresponding to (2.4) by ${}^{(l)}C^{a}_{\lambda\mu\nu}(a, b)$, $l = i, \ldots,$ vii and $C^{a}_{\lambda\mu\nu}(a)$. Thus

$$ {}^{(i)}C^{\rho}_{\lambda\mu\nu}(a, b) = {}^{a}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu} - {}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}, \tag{2.5a} $$

$$ {}^{(ii)}C^{\rho}_{\lambda\mu\nu}(a, b) = {}^{b}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu} - {}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\beta}_{\lambda\mu}, \tag{2.5b} $$

$$ {}^{(iii)}C^{\rho}_{\lambda\mu\nu}(a,b) = {}^{a}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu} - {}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\beta}_{\lambda\mu}, \tag{2.5c} $$

$$ {}^{(iv)}C^{\rho}_{\lambda\mu\nu}(a,b) = {}^{b}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu} - {}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}, \tag{2.5d} $$

$$ {}^{(v)}C^{\rho}_{\lambda\mu\nu}(a, b) = {}^{a}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu} - {}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}, \tag{2·5e} $$

$$ {}^{(vi)}C^{\rho}_{\lambda\mu\nu}(a, b) = {}^{b}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu} - {}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\beta}_{\lambda\mu}, \tag{2. 5f} $$

$$ {}^{(vii)}C^{\rho}_{\lambda\mu\nu}(a,b) = {}^{b}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu} - {}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}, \tag{2.5g} $$

$$ \text{and} \quad C^{\rho}_{\lambda\mu\nu}(a) = {}^{a}\Gamma^{\rho}_{\lambda\nu,\mu} - {}^{a}\Gamma^{\rho}_{\lambda\mu,\nu} + {}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu} - {}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\rho}_{\lambda\mu}. \tag{2.5 h} $$

The 8 other expressions are dented by ${}^{(l)}C^{\rho}_{\lambda\mu\nu}(b,a)$, $C^{\rho}_{\lambda\mu\nu}(b)$. As the indices can take values from 1 to n, each of these expressions is a geometrical entity of which $C^{\rho}_{\lambda\mu\nu}(a)$, $C^{\rho}_{\lambda\mu\nu}(\mathrm{b})$ are tensors, namely ordinary curvature tensors. They satisfy the following relations :

$$ \left.\begin{aligned} {}^{(i)}C^{\rho}_{\lambda\nu\mu}(b,a) + {}^{(ii)}C^{\rho}_{\lambda\mu\nu}(a,b) + {}^{(iii)}C^{\rho}_{\lambda\mu\nu}(b,a) &= {}^{(iv)}C^{\rho}_{\lambda\mu\nu}(a,b), \\ {}^{(iii)}C^{\rho}_{\lambda\nu\mu}(a,b) + {}^{(v)}C^{\rho}_{\lambda\mu\nu}(a,b) &= 0, \\ {}^{(iv)}C^{\rho}_{\lambda\nu\mu}(b,a) + {}^{(vi)}C^{\rho}_{\lambda\mu\nu}(a,b) &= 0, \\ {}^{(i)}C^{\rho}_{\lambda\nu\mu}(b,a) + {}^{(iii)}C^{\rho}_{\lambda\mu\nu}(b,a) + {}^{(vi)}C^{\rho}_{\lambda\mu\nu}(a,b) &= {}^{(vii)}C^{\rho}_{\lambda\mu\nu}(a,b), \\ {}^{(iii)}C^{\rho}_{\lambda\mu\nu}(a,b) - {}^{(iv)}C^{\rho}_{\lambda\mu\nu}(a,b) + {}^{(vii)}C^{\rho}_{\lambda\mu\nu}(a,b) &= C^{\rho}_{\lambda\mu\nu}(a) \end{aligned}\right\} \tag{2.6} $$

Therefore only the three entities (2. 5a) to (2. 5c) may be taken as independent. Further, corresponding to the 8 cases (2.4), the formula (2.1) reduces to the following :

$$v_{\substack{\lambda,\,\mu\nu\\ a\ r\\ b}}-v_{\substack{\lambda;\,\nu\mu\\ a\ s\\ b}}=v_a{}^{(\mathrm{i})}C^a{}_{\lambda\mu\nu}(a,b)+$$

$$+\,v_{a,\mu}\Big({}^a\Gamma^a_{\lambda\nu}-{}^b\Gamma^a_{\lambda\nu}\Big)+v_{a,\nu}\Big({}^b\Gamma^a_{\lambda\mu}-{}^a\Gamma^a_{\lambda\mu}\Big)-v_{\substack{\lambda;a\\ a}}\Big({}^r\Gamma^a_{\mu\nu}-{}^s\Gamma^a_{\nu\mu}\Big), \tag{2.7a}$$

$$v_{\substack{\lambda;\mu\nu\\ a\ r\\ b}}-v_{\substack{\lambda;\,\nu\mu\\ b\ s\\ a}}=v_a{}^{(\mathrm{ii})}C^a_{\lambda\mu\nu}(a,b)-v_{\substack{\lambda;a\\ a}}{}^r\Gamma^a_{\mu\nu}+v_{\substack{\lambda;a\\ b}}{}^s\Gamma^a_{\nu\mu}, \tag{2.7b}$$

$$v_{\substack{\lambda,\,\mu\nu\\ a\ r\\ b}}-v_{\substack{\lambda;\,\nu\mu\\ a\ s\\ a}}=v_a{}^{(\mathrm{iii})}C^a_{\lambda\mu\nu}(a,b)+$$

$$+\,v_{a,\mu}\Big({}^a\Gamma^a_{\lambda\nu}-{}^b\Gamma^a_{\lambda\nu}\Big)-v_{\substack{\lambda;a\\ a}}\Big({}^r\Gamma^a_{\mu\nu}-{}^s\Gamma^a_{\nu\mu}\Big), \tag{2.7c}$$

$$v_{\substack{\lambda,\mu\nu\\ a\ r\\ b}}-v_{\substack{\lambda;\nu\mu\\ b\ s\\ b}}=v_a{}^{(\mathrm{iv})}C^a_{\lambda\mu\nu}(a,b)+$$

$$+\,v_{a,\,\nu}\Big({}^b\Gamma^a_{\lambda\mu}-{}^a\Gamma^a_{\lambda\mu}\Big)-v_{\substack{\lambda;\,a\\ a}}{}^r\Gamma^a_{\mu\nu}+v_{\substack{\lambda;a\\ b}}{}^s\Gamma^a_{\nu\mu}, \tag{2.7d}$$

$$v_{\substack{\lambda;\mu\nu\\ a\ r\\ a}}-v_{\substack{\lambda;\nu\mu\\ a\ s\\ b}}=v_a{}^{(\mathrm{v})}C^a_{\lambda\mu\nu}(a,b)+$$

$$+\,v_{a,\nu}\Big({}^b\Gamma^a_{\lambda\mu}-{}^a\Gamma^a_{\lambda\mu}\Big)-v_{\substack{\lambda;a\\ a}}\Big({}^r\Gamma^a_{\mu\nu}-{}^s\Gamma^a_{\nu\mu}\Big), \tag{2.7e}$$

$$v_{\substack{\lambda,\mu\nu\\ a\ r\\ a}}-v_{\substack{\lambda;\nu\mu\\ b\ s\\ b}}=v_a{}^{(\mathrm{vi})}C^a_{\lambda\mu\nu}(a,b)+$$

$$+\,v_{a\,\mu}\Big({}^b\Gamma^a_{\lambda\nu}-{}^a\Gamma^a_{\lambda\nu}\Big)-v_{\substack{\lambda;\,a\\ a}}{}^r\Gamma^a_{\mu\nu}+v_{\substack{\lambda;a\\ b}}{}^s\Gamma^a_{\nu\mu}, \tag{2.7f}$$

$$v_{\substack{\lambda;\mu\nu\\ a\ r\\ a}}-v_{\substack{\lambda,\nu\mu\\ b\ s\\ b}}=v_a{}^{(\mathrm{vii})}C^a_{\lambda\mu\nu}(a,b)+$$

$$+\,v_{a,\mu}\Big({}^b\Gamma^a_{\lambda\nu}-{}^a\Gamma^a_{\lambda\nu}\Big)+v_{a,\nu}\Big({}^b\Gamma^a_{\lambda\mu}-{}^a\Gamma^a_{\lambda\mu}\Big)-v_{\substack{\lambda;a\\ a}}{}^r\Gamma^a_{\mu\nu}+v_{\substack{\lambda,a\\ b}}{}^s\Gamma^a_{\nu\mu}, \tag{2.7g}$$

$$v_{\substack{\lambda;\,\mu\nu\\ a\ r\\ a}}-v_{\substack{\lambda;\,\nu\mu\\ a\ s\\ a}}=v_a\,C^a_{\lambda\mu\nu}(a)-v_{\substack{\lambda,a\\ a}}\Big({}^r\Gamma^a_{\mu\nu}-{}^s\Gamma^a_{\nu\mu}\Big). \tag{2.7h}$$

By virtue of (2.6), only the first three of the above formulae are independent. The last one becomes the well-known Ricci formula when $r=s=a$.

3. Generalised curvature tensors

By means of the entities ${}^{(\mathrm{i})}C^{\rho}_{\lambda\mu\nu}(a, b)$, $C^{\rho}_{\lambda\mu\nu}(a)$ defined by (2.5a) to (2.5h) we now define new entities ${}^{m}R^{\rho}_{\lambda\mu\nu}(a, b)$, $m=1, 2, 3, 4$, as follows :

$$
{}^{1}R^{\rho}_{\lambda\mu\nu}(a,b) = \tfrac{1}{2}\Big({}^{(\mathrm{i})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{i})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big) = \tfrac{1}{2}\Big({}^{(\mathrm{ii})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{ii})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big)
$$

$$
=\tfrac{1}{2}\Big({}^{a}\Gamma^{\rho}_{\lambda\nu,\mu}+{}^{b}\Gamma^{\rho}_{\lambda\nu,\mu}-{}^{a}\Gamma^{\rho}_{\lambda\mu,\nu}-{}^{b}\Gamma^{\rho}_{\lambda\mu,\nu}+{}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu}+{}^{b}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu}-{}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}-{}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\beta}_{\lambda\mu}\Big) \qquad (3.1\mathrm{a})
$$

$$
{}^{2}R^{\rho}_{\lambda\mu\nu}(a,b)=\tfrac{1}{2}\Big({}^{(\mathrm{iii})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{iii})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big)=\tfrac{1}{2}\Big({}^{(\mathrm{iv})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{iv})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big)
$$

$$
=\tfrac{1}{2}\Big({}^{a}\Gamma^{\rho}_{\lambda\nu,\mu}+{}^{b}\Gamma^{\rho}_{\lambda\nu,\mu}-{}^{a}\Gamma^{\rho}_{\lambda\mu,\nu}-{}^{b}\Gamma^{\rho}_{\lambda\mu,\nu}+{}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu}+{}^{b}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu}-{}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\beta}_{\lambda\mu}-{}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}\Big), \quad (3.1b)
$$

$$
{}^{3}R^{\rho}_{\lambda\mu\nu}(a,b)=\tfrac{1}{2}\Big({}^{(\mathrm{v})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{v})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big)=\tfrac{1}{2}\Big({}^{(\mathrm{vi})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{vi})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big)
$$

$$
=\tfrac{1}{2}\Big({}^{a}\Gamma^{\rho}_{\lambda\nu,\mu}+{}^{b}\Gamma^{\rho}_{\lambda\mu,\nu}-{}^{a}\Gamma^{\rho}_{\lambda\mu,\nu}-{}^{b}\Gamma^{\rho}_{\lambda\mu,\nu}+{}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu}+{}^{b}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu}-{}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\beta}_{\lambda\mu}-{}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\beta}_{\lambda\mu}\Big), \quad (3.1\mathrm{c})
$$

$$
{}^{4}R^{\rho}_{\lambda\mu\nu}(a,b)=\tfrac{1}{2}\Big({}^{(\mathrm{vii})}C^{\rho}_{\lambda\mu\nu}(a,b)+{}^{(\mathrm{vii})}C^{\rho}_{\lambda\mu\nu}(b,a)\Big)=\tfrac{1}{2}\Big(C^{\rho}_{\lambda\mu\nu}(a)+C^{\rho}_{\lambda\mu\nu}(b)\Big)
$$

$$
=\tfrac{1}{2}\Big({}^{a}\Gamma^{\rho}_{\lambda\nu,\mu}+{}^{b}\Gamma^{\rho}_{\lambda\nu,\mu}-{}^{a}\Gamma^{\rho}_{\lambda\mu,\nu}-{}^{b}\Gamma^{\rho}_{\lambda\mu,\nu}+{}^{a}\Gamma^{\rho}_{\beta\mu}\,{}^{a}\Gamma^{\beta}_{\lambda\nu}+{}^{b}\Gamma^{\rho}_{\beta\mu}\,{}^{b}\Gamma^{\beta}_{\lambda\nu}-{}^{a}\Gamma^{\rho}_{\beta\nu}\,{}^{a}\Gamma^{\rho}_{\lambda\mu}-{}^{b}\Gamma^{\rho}_{\beta\nu}\,{}^{b}\Gamma^{\rho}_{\lambda\mu}\Big) \qquad (3.1d)
$$

By what has been said in the last sectio , only the first two of the new entities are independent. The last entity is half the sum of two curvature tensors. We shall however write

$$
C^{\rho}_{\lambda\mu\nu}(a) = R^{\rho}_{\lambda\mu\nu}(a). \quad \text{So } {}^{4}R^{\rho}_{\lambda\mu\nu}(a,b)=\tfrac{1}{2}\Big(R^{\rho}_{\lambda\mu\nu}(a)+R^{\rho}_{\lambda\mu\nu}(b)\Big). \qquad (3.2)
$$

If therefore in the formulae (2.7a) to (2.7f) we interchange a and b and take half the sum, we obtain the following three formulae :

$$
\tfrac{1}{2}\Big(\underset{\substack{a\ r\\ b}}{\nu}_{\lambda,\mu\nu}+\underset{\substack{b\ r\\ a}}{\nu}_{\lambda;\mu\nu}\Big)-\tfrac{1}{2}\Big(\underset{\substack{a\ s\\ b}}{\nu}_{\lambda,\nu\mu}+\underset{\substack{b\ s\\ a}}{\nu}_{\lambda;\nu\mu}\Big)
$$

$$
=\nu_{\alpha}\ {}^{1}R^{\alpha}_{\lambda\mu\nu}(a,b)-\tfrac{1}{2}\Big(\underset{a}{\nu}_{\lambda;\alpha}+\underset{b}{\nu}_{\lambda;\alpha}\Big)\Big({}^{r}\Gamma^{\alpha}_{\mu\nu}-{}^{s}\Gamma^{\alpha}_{\nu\mu}\Big). \qquad (3.3a)
$$

$$
\tfrac{1}{2}\Big(\underset{\substack{a\ r\\ b}}{\nu}_{\lambda,\mu\nu}+\underset{\substack{b\ r\\ a}}{\nu}_{\lambda;\mu\nu}\Big)-\tfrac{1}{2}\Big(\underset{\substack{a\ s\\ a}}{\nu}_{\lambda,\nu\mu}+\underset{\substack{b\ s\\ b}}{\nu}_{\lambda;\nu\mu}\Big)
$$

$$
=\nu_{\alpha}\ {}^{2}R^{\alpha}_{\lambda\mu\nu}(a,b)-\tfrac{1}{2}\Big(\underset{a}{\nu}_{\lambda;\alpha}+\underset{b}{\nu}_{\lambda,\alpha}\Big)\Big({}^{r}\Gamma^{\alpha}_{\mu\nu}-{}^{s}\Gamma^{\alpha}_{\nu\mu}\Big), \qquad (3.3b)
$$

$$\tfrac{1}{2}\Big(\underset{\substack{a\ r\\ a}}{v_{\lambda;\mu\nu}}+\underset{\substack{b\ r\\ b}}{v_{\lambda;\mu\nu}}\Big)-\tfrac{1}{2}\Big(\underset{\substack{a\ s\\ b}}{v_{\lambda;\nu\mu}}+\underset{\substack{b\ s\\ a}}{v_{\lambda;\nu\mu}}\Big)$$

$$=v_{a}\,{}^{3}R^{a}_{\lambda\mu\nu}(a,b)=\tfrac{1}{2}\Big(\underset{a}{v_{\lambda;a}}+\underset{b}{v_{\lambda,a}}\Big)\Big(\Gamma^{a}_{\mu\nu}-{}^{s}\Gamma^{a}_{\nu\mu}\Big)\,. \tag{3.3.c}$$

As said before, only the first two of these formulae are independent. Now it is at once seen from (3.3*a*) to (3.3*c*) that the entities

$${}^{1}R^{\rho}_{\lambda\mu\nu}(a,b)\,,\ {}^{2}R^{\rho}_{\lambda\mu\nu}(a,b),\ \ {}^{3}R^{\rho}_{\lambda\mu\nu}(a,b) \tag{3.4}$$

are tensors. As defined in (3. 1*a*) to (3.1*c*), they are symmetric in *a*, *b*, and satisfy the following properties

$${}^{1}R^{\rho}_{\lambda\mu\nu}(a,b)+{}^{1}R^{\rho}_{\lambda\nu\mu}(a,b)=0, \tag{3.5a}$$

$${}^{2}R^{\rho}_{\lambda\mu\nu}(a,b)+{}^{3}R^{\rho}_{\lambda\nu\mu}(a,b)=0. \tag{3.5b}$$

The tensors (3.4), of which the last two are not really distinct, will be called *generalised curvature tensors* (with respect to two affine connections). On account of the property (3.5b), which shows the nature of interdependence of the last two tensors, the curvature tensors ${}^{2}R^{\rho}_{\lambda\mu\nu}(a,b)$ and ${}^{3}R^{\rho}_{\lambda\mu\nu}(a,b)$ will be called *complementary* to one another. Applying this criterion, it follows from (3.5a) that the curvature tensor ${}^{1}R^{\rho}_{\lambda\mu\nu}(a,b)$ is *self-complementary* (*i.e.*, skew in the indices μ, ν). The following relations are seen to hold from definition (3. 1a) to (3. 1d) and notation (3.2),

$$\left.\begin{aligned}&{}^{1}R^{\rho}_{\lambda\mu\nu}(a,b)-{}^{2}R^{\rho}_{\lambda\mu\nu}(a,b)-{}^{3}R^{\rho}_{\lambda\mu\nu}(a,b)+{}^{4}R^{\rho}_{\lambda\mu\nu}(a,b)=0,\\ &\tfrac{1}{4}\Big[{}^{1}R^{\rho}_{\lambda\mu\nu}(a,b)+{}^{2}R^{\rho}_{\lambda\mu\nu}(a,b)+{}^{3}R^{\rho}_{\lambda\mu\nu}(a,b)+{}^{4}R^{\rho}_{\lambda\mu\nu}(a,b)\Big]=R^{\rho}_{\lambda\mu\nu}(c)\end{aligned}\right\} \tag{3·6}$$

where $${}^{c}\Gamma^{\rho}_{\lambda\mu}=\tfrac{1}{2}\Big({}^{a}\Gamma^{\rho}_{\lambda\mu}+{}^{b}\Gamma^{\rho}_{\lambda\mu}\Big).$$

As usual, the contracted generalised curvature tensors will be denoted as follows :

$${}^{m}R_{\lambda\mu}(a,b)={}^{m}R^{a}_{\lambda\mu a}(a,b),\ \ {}^{m}V_{\lambda\mu}(a,b)={}^{m}R^{a}_{a\lambda\mu}(a,b),\ m=1,2,3. \tag{3.7}$$

It therefore follows from (3.5b) that

$${}^{2}R_{\lambda\mu}(a,b)+{}^{3}R^{a}_{\lambda a\mu}(a,b)=0,\ {}^{3}R_{\lambda\mu}(a,b)+{}^{2}R^{a}_{\lambda a\mu}(a,b)=0,\ \ {}^{2}V_{\lambda\mu}(a,b)+{}^{3}V_{\mu\lambda}(a,b)=0. \tag{3.8}$$

4. Some generalised formulae

Let $R^{\rho}_{\lambda\mu\nu}$ be the ordinary curvature tensor formed with an arbitrary affine connection $\Gamma^{\rho}_{\lambda\mu}$, the covariant derivative with respect to which is denoted by a semi-colon. If $S^{\rho}_{\lambda\mu}=\frac{1}{2}\left(\Gamma^{\rho}_{\lambda\mu}-\Gamma^{\rho}_{\mu\lambda}\right)$, $S_{\mu}=S^{\alpha}_{\mu\alpha}$, then the following formulae are known :

(I) $R^{\rho}_{\lambda\mu\nu}+R^{\rho}_{\mu\nu\lambda}+R^{\rho}_{\nu\lambda\mu}$

$$=-2\left[S^{\rho}_{\nu\mu;\lambda}+S^{\rho}_{\nu\lambda;\mu}+S^{\rho}_{\lambda\mu;\nu}+2\left(S^{\rho}_{\lambda\alpha}S^{\alpha}_{\mu\nu}+S^{\rho}_{\mu\alpha}S^{\alpha}_{\nu\lambda}+S^{\rho}_{\nu\alpha}S^{\alpha}_{\lambda\mu}\right)\right]$$

(II) It follows that the contracted curvature tensors are related by

$$R_{\lambda\mu}-R_{\mu\lambda}+V_{\lambda\mu}=2\left(S_{\lambda,\mu}-S_{\mu,\lambda}-S^{\alpha}_{\lambda\mu;\alpha}\right)$$

(III) For any tensor $A_{\lambda_1\ldots\lambda m_1}$,

$$A_{\lambda_1\ldots\lambda m;\mu\nu}-A_{\lambda_1\ldots\lambda m;\nu\mu}$$

$$=\sum_{\alpha}^{1\ldots m}A_{\lambda_1,\ldots\lambda\alpha-1\,\rho\,\lambda\alpha+1,\ldots\lambda m}R^{\rho}_{\lambda_\alpha\mu\nu}-2\,A_{\lambda_1\ldots\lambda m;\alpha}S^{\alpha}_{\mu\nu}.$$

(IV) If $L^{\rho}_{\lambda\mu}=\Gamma^{\rho}_{\lambda\mu}+A^{\rho}_{\lambda\mu}$, where $A^{\rho}_{\lambda\mu}$ is a tensor, and $I^{\rho}_{\lambda\mu\nu}$ is the curvature tensor formed with $L^{\rho}_{\lambda\mu}$, then

$$L^{\rho}_{\lambda\mu\nu}=R^{\rho}_{\lambda\mu\nu}+A^{\rho}_{\lambda\nu;\mu}-A^{\rho}_{\lambda\mu;\nu}+A^{\rho}_{\alpha\mu}A^{\alpha}_{\lambda\nu}-A^{\rho}_{\alpha\nu}A^{\alpha}_{\lambda\mu}-2A^{\rho}_{\lambda\alpha}S^{\alpha}_{\mu\nu}.$$

(V) $R^{\rho}_{\lambda\mu\nu;\omega}+R^{\rho}_{\lambda\nu\omega;\mu}+R^{\rho}_{\lambda\omega\mu;\nu}$

$$=-2\left[R^{\rho}_{\lambda\mu\alpha}S^{\alpha}_{\nu\omega}+R^{\rho}_{\lambda\nu\alpha}S^{\alpha}_{\omega\mu}+R^{\rho}_{\lambda\omega\alpha}S^{\alpha}_{\mu\nu}\right].$$

We give below the corresponding formulae satisfied by the generalised curvature tensors. For this purpose we adopt the following notations :

$$\left.\begin{aligned}&{}^{a}S^{\rho}_{\lambda\mu}=\tfrac{1}{2}\left({}^{a}\Gamma^{\rho}_{\lambda\mu}-{}^{a}\Gamma^{\rho}_{\mu\lambda}\right),\ {}^{b}S^{\rho}_{\lambda\mu}=\tfrac{1}{2}\left({}^{b}\Gamma^{\rho}_{\lambda\mu}-{}^{b}\Gamma^{\rho}_{\mu\lambda}\right),\ T^{\rho}_{\lambda\mu}=\tfrac{1}{2}\left({}^{a}\Gamma^{\rho}_{\lambda\mu}-{}^{b}\Gamma^{\rho}_{\mu\lambda}\right),\\&\qquad{}^{a}S_{\lambda}={}^{a}S^{\alpha}_{\lambda\alpha},\ {}^{b}S_{\lambda}={}^{b}S^{\alpha}_{\lambda\alpha},\ T_{\lambda}=T^{\alpha}_{\lambda\alpha}.\end{aligned}\right\}\quad(4.1)$$

It follows : $T^{\rho}_{\lambda\mu} - T^{\rho}_{\mu\lambda} = {}^{a}S^{\rho}_{\lambda\mu} + {}^{b}S^{\rho}_{\lambda\mu}$, $T^{a}_{a\lambda} = T_{\lambda} - {}^{a}S_{\lambda} - {}^{b}S_{\lambda}$. (4.2)

Now it may be verified that corresponding to (I) we have

$$
\begin{aligned}
&{}^{1}R^{\rho}_{\lambda\mu\nu}(a,b) + {}^{1}R^{\rho}_{\mu\nu\lambda}(a,b) + {}^{1}R^{\rho}_{\nu\lambda\mu}(a,b) \\
&= -\Big[{}^{a}S^{\overset{\rho}{b}}_{\underset{bb}{\mu\nu;\lambda}} + {}^{a}S^{\overset{\rho}{b}}_{\underset{bb}{\nu\lambda;\mu}} + {}^{a}S^{\overset{\rho}{b}}_{\underset{bb}{\lambda\mu;\nu}} + {}^{b}S^{\overset{\rho}{a}}_{\underset{aa}{\mu\nu;\lambda}} + {}^{b}S^{\overset{\rho}{a}}_{\underset{aa}{\nu\lambda;\mu}} + {}^{b}S^{\overset{\rho}{a}}_{\underset{aa}{\lambda\mu;\nu}} \\
&\quad + 2\Big({}^{a}S^{\rho}_{\lambda a}\,{}^{b}S^{a}_{\mu\nu} + {}^{a}S^{\rho}_{\mu a}\,{}^{b}S^{a}_{\nu\lambda} + {}^{a}S^{\rho}_{\nu a}\,{}^{b}S^{a}_{\lambda\mu} + {}^{b}S^{\rho}_{\lambda a}\,{}^{a}S^{a}_{\mu\nu} + {}^{b}S^{\rho}_{\mu a}\,{}^{a}S^{a}_{\nu\lambda} + {}^{b}S^{\rho}_{\nu a}\,{}^{a}S^{a}_{\lambda\mu} \Big)\Big]
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
&{}^{2}R^{\rho}_{\lambda\mu\nu}(a,b) + {}^{2}R^{\rho}_{\quad\lambda}(a,b) + {}^{2}R^{\rho}_{\nu\lambda\mu}(a,b) \\
&= -\Big[T^{\overset{\rho}{a}}_{\underset{aa}{\mu\nu;\lambda}} + T^{\overset{\rho}{a}}_{\underset{aa}{\nu\lambda;\mu}} + T^{\overset{\rho}{a}}_{\underset{aa}{\lambda\mu,\nu}} - T^{\overset{\rho}{b}}_{\underset{bb}{\nu\mu;\lambda}} - T^{\overset{\rho}{b}}_{\underset{bb}{\lambda\nu;\mu}} - T^{\overset{\rho}{b}}_{\underset{bb}{\mu\lambda;\nu}} \\
&\quad + 2\Big(T^{\rho}_{\lambda a}T^{a}_{\mu\nu} + T^{\rho}_{\mu a}T^{\sigma}_{\nu\lambda} + T^{\rho}_{\nu a}T^{a}_{\lambda\mu} + T^{\rho}_{a\lambda}T^{a}_{\nu\mu} + T^{\rho}_{a\mu}T^{a}_{\lambda\nu} + T^{\rho}_{a\nu}T^{a}_{\mu\lambda} \Big)\Big]
\end{aligned}
\tag{4.4}
$$

Similarly for the cyclic sum of ${}^{3}R^{\rho}_{\lambda\mu\nu}$ (a, b) and the sum of the above two cycles. These are generalisations of cyclic property of curvature tensor. Corresponding to (II), the contracted generalised curvature tensors satisfy

$$
\begin{aligned}
&{}^{1}R_{\lambda\mu}(a,b) - {}^{1}R_{\mu\lambda}(a,b) + {}^{1}V_{\lambda\mu}(a,b) \\
&= \left({}^{a}S_{\lambda} + {}^{b}S_{\lambda}\right)_{,\mu} - \left({}^{a}S_{\mu} + {}^{b}S_{\mu}\right)_{,\lambda} - \Big({}^{a}S^{\overset{a}{b}}_{\underset{bb}{\lambda\mu}} + {}^{b}S^{\overset{a}{a}}_{\underset{aa}{\lambda\mu}}\Big)_{;a}\,.
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
&\tfrac{1}{2}\left({}^{2}R_{\lambda\mu}(a,b) + {}^{3}R_{\lambda\mu}(a,b)\right) - \tfrac{1}{2}\left({}^{2}R_{\mu\lambda}(a,b) + {}^{3}R_{\mu\lambda}(a,b)\right) + \tfrac{1}{2}\left({}^{2}V_{\lambda\mu}(a,b) + {}^{3}V_{\lambda\mu}(a,b)\right) \\
&= \left({}^{a}S_{\lambda} + {}^{b}S_{\lambda}\right)_{,\mu} - \left({}^{a}S_{\mu} + {}^{b}S_{\mu}\right)_{,\lambda} - \tfrac{1}{2}\Big({}^{a}S^{\overset{a}{a}}_{\underset{aa}{\lambda\mu}} + {}^{a}S^{\overset{a}{b}}_{\underset{bb}{\lambda\mu}} + {}^{b}S^{\overset{a}{a}}_{\underset{aa}{\lambda\mu}} + {}^{b}S^{\overset{a}{b}}_{\underset{bb}{\lambda\mu}}\Big)_{;a}\,.
\end{aligned}
\tag{4.6}
$$

Corresponding to (III) we have the following generalised formulae :

$$
\begin{aligned}
&A_{\underset{\underset{a_1\ldots a_m}{a_1\ldots a_m}}{\lambda_1\ldots\ldots\lambda_m}\,;\,\underset{c}{\mu\nu}} - A_{\underset{\underset{a_1\ldots a_m}{a_1\ldots a_m}}{\lambda_1\ldots\lambda_m}\,;\,\underset{d}{\nu\mu}} \\
&= \sum_{a}^{1..m} A_{\lambda_1\ldots\lambda_{a-1}\,\rho\,\lambda_{a+1}\ldots\lambda_m}\, R^{\rho}_{\lambda_a\mu\nu}(a) - A_{\underset{a_1\ldots a_m}{\lambda_1\ldots\lambda_m}\,;\,\rho}\left({}^{c}\Gamma^{\rho}_{\mu\nu} - {}^{d}\Gamma^{\rho}_{\nu\mu}\right).
\end{aligned}
\tag{4.7}
$$

$$\tfrac{1}{2}\Big(A_{\substack{\lambda_1\,.\,\lambda_m\,;\mu\nu\\ a_1\,..a_m\ c\\ b_1\,.\,b_m}}+A_{\substack{\lambda_1\,.\,\lambda_m\,;\mu\nu\\ b_1\,.\,b_m\ c\\ a_1..\,a_m}}\Big)-\tfrac{1}{2}\Big(A_{\substack{\lambda_1\ldots\lambda_m\,;\nu\mu\\ a_1\,.a_m\ d\\ b_1\ldots b_m}}+A_{\substack{\lambda_1\ldots\lambda_m\,;\nu\mu\\ b_1\,..b_m\ d\\ a_1\ldots a_m}}\Big)$$

$$=\sum_{a}^{1\,.\,m} A_{\lambda_1\,.\,\lambda_{a-1}\rho\,\lambda_{a+1}..\,\lambda_m}\,{}^1R^{\rho}_{\lambda_a\,\mu\nu}(a_a,b_a)$$

$$-\tfrac{1}{2}\Big(A_{\substack{\lambda_1\ \ \lambda_m;\rho\\ a_1.\ a_m}}+A_{\substack{\lambda_1\ .\lambda_m;\rho\\ b_1\,.\,b_m}}\Big)\Big({}^c\Gamma^{\rho}_{\mu\nu}-{}^d\Gamma^{\rho}_{\nu\mu}\Big). \tag{4.8}$$

This is the generalisation of (3.3a). It is a generalisation of the Ricci formula.

Also
$$\tfrac{1}{2}\Big(A_{\substack{\lambda_1\lambda_2;\,\mu\nu\\ a_1a_2\ \ c\\ b_1b_2}}+A_{\substack{\lambda_1\lambda_2;\,\mu\nu\\ b_1b_2\ \ c\\ a_1a_2}}\Big)-\tfrac{1}{2}\Big(A_{\substack{\lambda_1\lambda_2\,;\nu\mu\\ a_1b_2\ \ d\\ a_1b_2}}+A_{\substack{\lambda_1\lambda_2\,;\nu\mu\\ b_1a_2\ \ d\\ b_1a_2}}\Big)$$

$$=A_{\rho\lambda_2}\,{}^2R^{\rho}_{\lambda_1\mu\nu}(a_1,b_1)+A_{\lambda_1\rho}\,{}^2R^{\rho}_{\lambda_2\mu\nu}(a_1,b_1)$$

$$-\tfrac{1}{2}\Big(A_{\substack{\lambda_1\lambda_2;\rho\\ a_1\,a_2}}+A_{\substack{\lambda_1\lambda_2;\rho\\ b_1b_2}}\Big)\Big({}^c\Gamma^{\rho}_{\mu\nu}-{}^d\Gamma^{\rho}_{\nu\mu}\Big) \tag{4.9}$$

Again
$$\tfrac{1}{2}\Big(A_{\substack{\lambda_1\ldots\ \ldots\lambda_m;\mu\nu\\ a_1a_2a_3\,..a_m\ c\\ b_1b_2a_3\,.\,a_m}}+A_{\substack{\lambda_1\ldots\ldots\,.\lambda_m;\mu\nu\\ b_1b_2a_3\ldots a_m\ c\\ a_1a_2a_3\,.\,a_m}}\Big)-\tfrac{1}{2}\Big(A_{\substack{\lambda_1\,..\ \ .\ \lambda_m;\nu\mu\\ a_1b_2a_3\ \ a_md\\ a_1b_2a_3\ \ a_m}}+A_{\substack{\lambda_1..\ .\ ..\ \lambda_m;\nu\mu\\ b_1a_2a_3\ \ a_md\\ b_2a_2a_3.\ .a_m}}\Big)$$

$$=A_{\rho\lambda_2\cdots\lambda_m}\,{}^2R^{\rho}_{\lambda_1\mu\nu}(a_1,b_1)+A_{\lambda_1\rho\lambda_3\cdot\cdot\,\lambda_m}\,{}^2R^{\rho}_{\lambda_2\mu\nu}(a_2,b_2)$$

$$+\sum_{a}^{3\,.\,m}A_{\lambda_1\ \ldots\lambda_{a-1}\rho\lambda_{a+1}.\,\ldots\,\lambda_m}\,R^{\rho}{}_{\lambda_a\mu\nu}(a_a)-\tfrac{1}{2}\Big(A_{\substack{\lambda_1\ldots.\,.\lambda_m;\rho\\ a_1\ldots\ldots\,a_m}}+A_{\substack{\lambda_1\ .\ \ldots\ \lambda_m;\rho\\ b_1b_2a_3\ \ a_m}}\Big)\Big(c\Gamma^{\rho}_{\mu\nu}-d\Gamma^{\rho}_{\nu\mu}\Big). \tag{4.10}$$

This is a generalisation of (3.3b). It is also a generalisation of Ricci formula.

Corresponding to (IV) the generalised formula is the following :

If ${}^a\Gamma^{\rho}_{\lambda\mu}=\Gamma^{\rho}_{\lambda\mu}+A^{\rho}_{\lambda\mu}$, ${}^b\Gamma^{\rho}_{\lambda\mu}=\Gamma^{\rho}_{\lambda\mu}+B^{\rho}_{\lambda\mu}$, then

$${}^1R^{\rho}_{\lambda\mu\nu}(a_1,b)=R^{\rho}_{\lambda\mu\nu}+\tfrac{1}{2}\Big[\Big(A^{\rho}_{\lambda\nu}+B^{\rho}_{\lambda\nu}\Big)_{;\,\mu}-\Big(A^{\rho}_{\lambda\mu}+B^{\rho}_{\lambda\mu}\Big)_{;\,\nu}$$

$$+A^{\rho}_{\alpha\mu}B^{\alpha}_{\lambda\nu}+B^{\rho}_{\alpha\mu}A^{\alpha}_{\lambda\nu}-A^{\rho}_{\alpha\nu}B^{\alpha}_{\lambda\mu}-B^{\rho}_{\alpha\nu}A^{\alpha}_{\lambda\mu}\Big]-\Big(A^{\rho}_{\lambda\alpha}+B^{\rho}_{\lambda\alpha}\Big)S^{\alpha}_{\mu\nu}. \tag{4.11}$$

Corresponding to (V) we have

$${}^{(i)}C^{\substack{\rho\\ b}}_{\substack{\lambda\mu\nu\\ b\,b\,b}}(a,b)_{;\,\omega}+{}^{(i)}C^{\substack{\rho\\ b}}_{\substack{\lambda\nu\omega\\ b\,b\,b}}(a,b)_{;\,\mu}+{}^{(i)}C^{\substack{\rho\\ b}}_{\substack{\lambda\omega\mu\\ b\,b\,b}}(a,b)_{;\nu}+{}^{(i)}C^{\substack{\rho\\ a}}_{\substack{\lambda\mu\nu\\ a\,a\,a}}(b,a)_{,\,\omega}+{}^{(i)}C^{\substack{\rho\\ a}}_{\substack{\lambda\nu\omega\\ a\,a}}(a,b)_{;\mu}$$

$$+{}^{(i)}C^{\substack{\rho\\ a}}_{\substack{\lambda\omega\mu\\ a\,a\,a}}(b,a)_{;\nu}=$$

$$
\begin{aligned}
&= -2\Big[{}^{(1)}C^{\rho}_{\lambda\mu a}(a,b)\, {}^{b}S_{\nu\omega} + {}^{(1)}C^{\rho}_{\lambda\nu a}(a,b)\, {}^{b}S^{a}_{\omega\mu} + {}^{(1)}C^{\rho}_{\lambda\omega a}(a,b)\, {}^{b}S^{a}_{\mu\nu} \\
&\qquad + {}^{(1)}C^{\rho}_{\lambda\mu a}(b,a)\, {}^{a}S^{a}_{\nu\omega} + {}^{(1)}C^{\rho}_{\lambda\nu a}(b,a)\, {}^{a}S^{a}_{\omega\mu} + {}^{(1)}C^{\rho}_{\lambda\omega a}(b,a)\, {}^{a}S^{a}_{\mu\nu}\Big] \\
&+ W^{\rho}_{a\nu\omega}(a,b)\left({}^{a}\Gamma^{a}_{\lambda\mu} - {}^{b}\Gamma^{a}_{\lambda\mu}\right) + W^{\rho}_{a\omega\mu}(a,b)\left({}^{a}\Gamma^{a}_{\lambda\nu} - {}^{b}\Gamma^{a}_{\lambda\nu}\right) + W^{\rho}_{a\mu\nu}(a,b)\left({}^{a}\Gamma^{a}_{\lambda\omega} - {}^{b}\Gamma^{a}_{\lambda\omega}\right), \qquad (4.12)
\end{aligned}
$$

where $W^{\rho}_{\lambda\mu\nu}(a,b) = {}^{a}\Gamma^{\rho}_{\beta\mu}\, {}^{b}\Gamma^{\beta}_{\lambda\nu} - {}^{b}\Gamma^{\rho}_{\beta\mu}\, {}^{a}\Gamma^{\beta}_{\lambda\nu} + {}^{b}\Gamma^{\rho}_{\beta\nu}\, {}^{a}\Gamma^{\beta}_{\lambda\mu} - {}^{a}\Gamma^{\rho}_{\beta\nu}\, {}^{b}\Gamma^{\beta}_{\lambda\mu}$.

So $W^{\rho}_{\lambda\mu\nu}(a,b)$ is skew in μ, ν as well as in a, b. It must however be noticed that (4.12) is not a tensorial relation in the usual sense unless $a = b$ in which case it reduces to (V).

5. Space governed by certain equations

Let there exist in our n-dimensional space a non-singular asymmetric tensor $g_{\lambda\mu}$ whose symmetric and skew-symmetric parts are denoted by $h_{\lambda\mu}$ and $k_{\lambda\mu}$ respectively. Also let the tensor $h_{\lambda\mu}$ be non-singular, $\left\{ {\rho \atop \lambda\mu} \right\}$ denote the Christoffel symbol formed by it and ${}^{a}\Gamma^{\rho}_{\lambda\mu}$, ${}^{b}\Gamma^{\rho}_{\lambda\mu}$, be two asymmetric connections. Now suppose the space satisfies the property

$$
g_{\underset{ab'}{\lambda\mu;\nu}} = 0, \quad \text{where} \quad {}^{b'}\Gamma^{\rho}_{\lambda\mu} = {}^{b}\Gamma^{\rho}_{\mu\lambda}. \qquad (5.1)
$$

On the assumption that the quantities $g_{\lambda\mu}$ are given and with reference to the notations (4.1), it follows from what was shown in a previous paper (Sen, 1964) that if the equations (5.1) admit a solution $({}^{a}\Gamma^{\rho}_{\lambda\mu}, {}^{b}\Gamma^{\rho}_{\lambda\mu})$, then this solution must be of the form

$$
{}^{a}\Gamma^{\rho}_{\lambda\mu} = \left\{ {\rho \atop \lambda\mu} \right\} + {}^{a}S^{\rho}_{\lambda\mu} + U^{\rho}_{\lambda\mu}, \qquad {}^{b}\Gamma^{\rho}_{\lambda\mu} = \left\{ {\rho \atop \lambda\mu} \right\} + {}^{b}S^{\rho}_{\lambda\mu} + V^{\rho}_{\lambda\mu}, \qquad (5.2)
$$

where $U^{\rho}_{\lambda\mu} = h^{\rho a}\left\{ 2h_{\beta(\lambda}\, {}^{a}S_{\mu)a} + g_{\lambda\beta} T^{\beta}_{[a\mu]} + g_{\mu\beta} T^{\beta}_{[a\lambda]} + g_{a\beta} T^{\beta}_{(\lambda\mu)} \right\}$,

$$
V^{\rho}_{\lambda\mu} = h^{\rho a}\left\{ -2h_{\beta(\lambda}\, {}^{b}S_{\mu)a} + g_{\beta\lambda} T^{\beta}_{[\mu a]} + g_{\beta\mu} T^{\beta}_{[\lambda a]} - g_{\beta a} T^{\beta}_{(\lambda\mu)} \right\}. \qquad (5.2a)
$$

The equations (5.2) show that if ${}^{a}S^{\rho}_{\lambda\mu}$, ${}^{b}S^{\rho}_{\lambda\mu}$, $T^{\rho}_{\lambda\mu}$ were known, the solution of (5.1) could be obtained. Also from (5.2) we get

$$
\left\{ {\rho \atop \lambda\mu} \right\} = {}^{a}\Gamma^{\rho}_{(\lambda\mu)} + {}^{b}\Gamma^{\rho}_{(\lambda\mu)} + h^{\rho a}\left\{ h_{\beta(\lambda}\left({}^{b}S^{\beta}_{\mu)a} - {}^{a}S^{\beta}_{\mu)a} \right) + k_{\lambda\beta} T^{\beta}_{[\mu a]} + k_{\mu\beta} T^{\beta}_{[\lambda a]} + k_{\beta a} T^{\beta}_{(\lambda\mu)} \right\}. \qquad (5.3)
$$

In this section we propose to establish some theorems involving the first generalised curvature tensor, its contracted curvature tensors and ${}^{a}S^{\rho}_{\lambda\mu}$, ${}^{b}S^{\rho}_{\lambda\mu}$, $T^{\rho}_{\lambda\mu}$ on the supposition that the equations (5.1) are satisfied. It follows from (5.1) that

$$
g_{\underset{aa}{\lambda\mu,\nu}} = -2g_{\lambda a} T^{a}_{\mu\nu}, \quad g_{\underset{b'b'}{\lambda\mu;\nu}} = 2g_{a\mu} T^{a}_{\lambda\nu} \qquad (5.4)
$$

So $$g_{\substack{\lambda\mu,\nu\omega\\ a\,a\,a\\ b'b'}} = -2\Big(g_{\substack{\lambda\alpha\\ b'b'}} T^{\substack{\alpha\\ b'}}_{\substack{\mu\nu\\ b'a}}\Big)_{;\omega} = -2\Big\{g_{\lambda\alpha} T^{\substack{\alpha\\ b'}}_{\substack{\mu\nu;\omega\\ b'a}} + 2g_{\beta\alpha} T^{\beta}_{\lambda\omega} T^{\alpha}_{\mu\nu}\Big\},$$

$$g_{\substack{\lambda\mu;\nu\omega\\ b'b'a\\ a\,a}} = 2\Big(g_{\substack{\alpha\mu\\ a\,a}} T^{\substack{\alpha\\ a}}_{\substack{\lambda\nu\\ a\,a}}\Big)_{;\omega} = 2\Big\{g_{\;\mu} T^{\substack{\alpha\\ a}}_{\substack{\lambda\nu;\omega\\ a\,a}} - 2g_{\alpha\beta} T^{\alpha}_{\lambda\nu} T^{\beta}_{\mu\omega}\Big\}$$

Therefore

$$\tfrac{1}{2}\Big(g_{\substack{\lambda\mu;\nu\omega\\ a\,a\,a\\ b'b'}} + g_{\substack{\lambda\mu;\nu\omega\\ b'b'a\\ a\,a}}\Big) - \tfrac{1}{2}\Big(g_{\substack{\lambda\mu,\omega\nu\\ a\,a\,b\\ b'b'}} + g_{\substack{\lambda\mu;\omega\nu\\ b'b'b\\ a\,a}}\Big)$$

$$= g_{\alpha\mu}\Big(T^{\substack{\alpha\\ a}}_{\substack{\lambda\nu;\omega\\ a\,a}} - T^{\substack{\alpha\\ a}}_{\substack{\lambda\omega;\nu\\ a\,b}}\Big) - g_{\lambda\alpha}\Big(T^{\substack{\alpha\\ b'}}_{\substack{\mu\nu;\omega\\ b'a}} - T^{\substack{\alpha\\ b'}}_{\substack{\mu\omega;\nu\\ b'b'}}\Big)$$

The left-hand side is, by (4.8),

$$g_{\mu\alpha}\,{}^{1}R^{\alpha}_{\lambda\nu\omega}(a,b') + g_{\lambda\alpha}\,{}^{1}R^{\alpha}_{\mu\nu\omega}(a,b') - \tfrac{1}{2}\Big(g_{\substack{\lambda\mu;a\\ a\,a}} + g_{\substack{\lambda\mu;a\\ b'b'}}\Big)T^{a}_{\nu\omega}$$

Therefore applying (5.4) again, we get

$$g_{\alpha\mu}\,{}^{1}R^{\alpha}_{\lambda\nu\omega}(a,b') + g_{\lambda\alpha}\,{}^{1}R^{\alpha}_{\mu\nu\omega}(a,b') + \Big(g_{\lambda\beta}T^{\beta}_{\mu a} - g_{\beta\mu}T^{\beta}_{\lambda a}\Big)T^{a}_{\nu\omega}$$

$$= g_{\;\mu}\Big(T^{\substack{\alpha\\ a}}_{\substack{\lambda\nu;\omega\\ a\,a}} - T^{\substack{\alpha\\ a}}_{\substack{\lambda\omega,\nu\\ a\,b}}\Big) - g_{\lambda\alpha}\Big(T^{\substack{\alpha\\ b'}}_{\substack{\mu\nu;\omega\\ b'a}} - T^{\substack{\alpha\\ b'}}_{\substack{\mu\omega;\nu\\ b'b}}\Big)$$

This can be written as

$$\Big[\delta^{\beta}_{\mu}\Big\{{}^{1}R^{\alpha}_{\lambda\nu\omega}(a,b') - T^{\alpha}_{\lambda\gamma}T^{\gamma}_{\nu\omega} - \Big(T^{\substack{\alpha\\ a}}_{\substack{\lambda\nu;\omega\\ a\,a}} - T^{\substack{\alpha\\ a}}_{\substack{\lambda\omega;\nu\\ a\,b}}\Big)\Big\}$$

$$+\delta^{\alpha}_{\lambda}\Big\{{}^{1}R^{\beta}_{\mu\nu\omega}(a,b') + T^{\beta}_{\mu\gamma}T^{\gamma}_{\nu\omega} + \Big(T^{\substack{\beta\\ b'}}_{\substack{\mu\nu;\omega\\ b'a}} - T^{\substack{\beta\\ b'}}_{\substack{\mu\omega;\nu\\ b'b}}\Big)\Big\}\Big]g_{\alpha\beta} = 0. \tag{5.5}$$

Hence we have the following theorem :

Theorem 1. *If the equations (5.1) hold, then the equations (5.5) also hold.*

Note. The equations (5.5) can be so expressed that instead of the symbol b' it shall have the symbol b. The result however will be big enough.

As usual, denote $g^{\lambda\mu} = g^{-1}\dfrac{\partial g}{\partial g_{\lambda\mu}}$, where $g=\det.\,\left|g_{\lambda\mu}\right|$.

So, $g^{\lambda\alpha}\,g_{\nu\alpha} = \delta^{\lambda}_{\nu}$, $g^{\alpha\mu}\,g_{\alpha\nu} = \delta^{\mu}_{\nu}$.

Then from the first of the equations (5.4) we get

$$g^{\lambda\mu}\underset{a\,a}{g}_{\lambda\mu;\nu} = -2T^{\alpha}_{\alpha\nu}, \quad \text{or} \quad g_{,\nu} - 2\,{}^{a}\Gamma^{\alpha}_{\alpha\nu}g = -2\,T^{\alpha}_{\varrho\sigma}g.$$

Similarly from the second equation we get

$$g_{,\nu} - 2\,{}^{b'}\Gamma^{\alpha}_{\alpha\nu}\,g = 2\,T^{\alpha}_{\alpha\nu}\,g\,.$$

Let the left-hand sides of the above two equations be denoted by $\underset{a}{g}_{;\nu}$ and $\underset{b'}{g}_{;\nu}$. Therefore writing $\partial_\nu g$ for $g_{,\nu}$ and applying (4.2) we have

$$\left.\begin{aligned} \underset{a}{g}_{;\nu} &= \left(\partial_\nu - 2\,{}^{a}\Gamma^{\alpha}_{\alpha\nu}\right)g \quad \text{and so} \quad \underset{a}{g}_{;\nu} = 2\left({}^{a}S_\nu + {}^{b}S_\nu - T_\nu\right)g\,; \\ \underset{b'}{g}_{,\nu} &= \left(\partial_\nu - 2\,{}^{b'}\Gamma^{\alpha}_{\alpha\nu}\right)g \quad \text{and so} \quad \underset{b'}{g}_{;\nu} = 2\left(T_\nu - {}^{a}S_\nu - {}^{b}S_\nu\right)g\,; \end{aligned}\right\} \tag{5.6}$$

The quantities $\underset{a}{g}_{;\nu}$, $\underset{b'}{g}_{;\nu}$ are to be regarded as covariant derivatives of g and so

$$\underset{a\ b'}{g}_{;\nu\omega} = \left(\partial_\omega\partial_\nu - 2\,{}^{a}\Gamma^{\alpha}_{\alpha\nu,\omega}\right)g - \underset{a}{g}_{;\beta}\,{}^{b'}\Gamma^{\beta}_{\nu\omega}g\,.$$

Therefore $\underset{a\ b'}{g}_{,\nu\omega} - \underset{a\ b'}{g}_{;\omega\nu} = 2\left({}^{a}\Gamma^{\alpha}_{\alpha\omega,\nu} - {}^{a}\Gamma^{\alpha}_{\alpha\nu,\omega}\right)g + 2\,\underset{a}{g}_{,\beta}\,{}^{b'}S^{\beta}_{\omega\nu}\,g$

Accordingly $\frac{1}{2}\left(\underset{a\ b'}{g}_{,\nu\omega} + \underset{b'\ a}{g}_{;\nu\omega}\right) - \frac{1}{2}\left(\underset{a\ b'}{g}_{;\omega\nu} + \underset{b'\ a}{g}_{;\omega\nu}\right)$

$$= \left({}^{a}\Gamma^{\alpha}_{\alpha\omega,\nu} - {}^{a}\Gamma^{\alpha}_{\alpha\nu,\omega} + {}^{b'}\Gamma^{\alpha}_{\alpha\omega,\nu} - {}^{b'}\Gamma^{\alpha}_{\alpha\nu,\omega}\right)g + \left(\underset{a}{g}_{;\beta}\,{}^{b'}S^{\beta}_{\omega\nu} + \underset{b'}{g}_{;\beta}\,{}^{a}S^{\beta}_{\omega\nu}\right)g$$

$$= 2\left[{}^{1}R_{\nu\omega}(a,b') + \left(T_\beta - {}^{a}S_\beta - {}^{b}S_\beta\right)\left({}^{a}S^{\beta}_{\omega\nu} + {}^{b}S^{\beta}_{\omega\nu}\right)\right], \quad \text{by (4.1) and (5.6).}$$

Also from (5.6) we get

$$\underset{a\ b'}{g}_{;\nu\omega} = 2\left\{\left({}^{a}S_\nu + {}^{b}S_\nu - T_\nu\right)_{,\omega} - \left({}^{a}S_\beta + {}^{b}S_\beta - T_\beta\right){}^{b'}\Gamma^{\beta}_{\nu\omega}\right\}g$$

Similarly for $\underset{b'\ a}{g}_{;\nu\omega}$. Therefore

$$\frac{1}{2}\left(\underset{a\ b'}{g}_{,\nu\omega} + \underset{b'\ a}{g}_{;\nu\omega}\right) - \frac{1}{2}\left(\underset{a\ b'}{g}_{;\omega\nu} + \underset{b'\ a}{g}_{;\omega\nu}\right)$$

$$= 2\left({}^{a}S_\beta + {}^{b}S_\beta - T_\beta\right)\left({}^{a}S^{\beta}_{\nu\omega} + {}^{b}S^{\beta}_{\nu\omega}\right)g$$

Equating the two, we obtain

$$\left.\begin{aligned} &{}^1V_{\upsilon\omega}(a,b') = 0 \\ \text{or} \qquad &{}^1V_{\upsilon\omega}(a,b) = 2\,{}^bS_{[\upsilon,\omega]} \end{aligned}\right\} \tag{5.7}$$

Accordingly (4.5) gives

$$2\,{}^1R_{[\lambda\mu]}(a,b) = 2\,{}^aS_{[\lambda,\mu]} - \Big({}^aS^{\overset{a}{b}}_{\underset{bb}{\lambda\mu}} + {}^bS^{\overset{a}{a}}_{\underset{aa}{\lambda\mu}}\Big)_{;\,a} \tag{5.8}$$

Hence we may state the following theorem

Theorem 2. *If the equations (5.1) hold, then the equations (5.7) and (5.8) also hold.*

Again, the equations (5.6) can be written as

$$ {}^a\Gamma^{\alpha}_{\alpha\nu} = \tfrac{1}{2}\big(I_n g\big)_{,\,\nu} + \big(T_\nu - {}^aS_\nu - {}^bS_\nu\big)$$

$$ {}^b\Gamma^{\alpha}_{\nu\alpha} = \tfrac{1}{2}\big(I_n g\big)_{,\,\nu} - \big(T_\nu - {}^aS_\nu - {}^bS_\nu\big)$$

Adding $2\,{}^aS_\nu$ to both sides of the first equation and $-2\,{}^bS_\nu$ to both sides of the second, we get

$$ {}^a\Gamma^{\alpha}_{\nu\alpha} = \tfrac{1}{2}\big(I_n g\big)_{,\,\nu} + \big(T_\nu + {}^aS_\nu - {}^bS_\nu\big).$$

$$ {}^b\Gamma^{\alpha}_{\alpha\nu} = \tfrac{1}{2}\big(I_n g\big)_{,\,\nu} - \big(T_\nu - {}^aS_\nu + {}^bS_\nu\big).$$

Therefore

$$\left.\begin{aligned} &{}^a\Gamma^{\alpha}_{\alpha\nu} + {}^b\Gamma^{\alpha}_{\nu\alpha} = \big(I_n g\big)_{,\,\nu} \\ &{}^a\Gamma^{\alpha}_{\nu\alpha} + {}^b\Gamma^{\alpha}_{\nu\alpha} = \big(I_n g\big)_{,\,\nu} + 2\,{}^aS_\nu \end{aligned}\right\} \tag{5.9}$$

Now put

$$\begin{aligned} P_{\lambda\mu}(a,b) = {} & \tfrac{1}{2}\Big({}^a\Gamma^{\alpha}_{\lambda\mu,\alpha} + {}^b\Gamma^{\alpha}_{\lambda\mu,\alpha} - {}^a\Gamma^{\alpha}_{\beta\mu}\,{}^b\Gamma^{\beta}_{\lambda\alpha} - {}^b\Gamma^{\alpha}_{\beta\mu}\,{}^a\Gamma^{\beta}_{\lambda\alpha}\Big) \\ & - \tfrac{1}{4}\Big\{{}^a\Gamma^{\alpha}_{\lambda\alpha,\mu} + {}^b\Gamma^{\alpha}_{\alpha\lambda,\mu} + {}^a\Gamma^{\alpha}_{\alpha\mu,\lambda} + {}^b\Gamma^{\alpha}_{\mu\alpha,\lambda} \\ & - \big({}^a\Gamma^{\alpha}_{\beta\alpha} + {}^a\Gamma^{\alpha}_{\alpha\beta}\big)\,{}^b\Gamma^{\beta}_{\lambda\mu} - \big({}^b\Gamma^{\alpha}_{\beta\alpha} + {}^b\Gamma^{\alpha}_{\alpha\beta}\big)\,{}^a\Gamma^{\beta}_{\lambda\mu}\Big\} \end{aligned} \tag{5.10}$$

Then it may be seen that

$$\begin{aligned} & {}^1R_{\lambda\mu}(a,b) + P_{\lambda\mu}(a,b) \\ = {} & \tfrac{1}{2}\Big({}^aS_{\underset{b}{\lambda};\,\mu} + {}^bS_{\underset{a}{\lambda},\,\mu}\Big) - \tfrac{1}{2}\,{}^aS_{\lambda,\mu} + \tfrac{1}{4}\Big({}^a\Gamma^{\alpha}_{\lambda\alpha,\mu} + {}^b\Gamma^{\alpha}_{\lambda\alpha,\mu}\Big) - \tfrac{1}{4}\Big({}^a\Gamma^{\alpha}_{\alpha\mu,\lambda} + {}^b\Gamma^{\alpha}_{\mu\alpha,\lambda}\Big). \end{aligned}$$

Substituting from (5.9) we ultimately get

$$ {}^{1}R_{\lambda\mu}(a,b)+P_{\lambda\mu}(a,b)=\tfrac{1}{2}\left({}^{a}S_{\underset{b}{\lambda}}+{}^{b}S_{\underset{a}{\lambda}}\right)_{,\mu} \tag{5.11}$$

Hence we may state the following theorem :

Theorem 3. *If the equations (5.1) hold, then the equations (5.11) also hold, where* $P_{\lambda\mu}$ *(a,b) is a tensor defined by (5.10).*

Finally, if in (4.11) we write $\Gamma^{\rho}_{\lambda\mu}=\left\{{\rho \atop \lambda\mu}\right\}$, $R^{\rho}_{\lambda\mu\nu}=H^{\rho}_{\lambda\mu\nu}$

and denote the covariant derivative with respect to $\left\{{\rho \atop \lambda\mu}\right\}$ by a crooked bracket followed by indices, we obtain

$$ {}^{1}R^{\rho}_{\lambda\mu\upsilon}(a,b)=H^{\rho}_{\lambda\mu\upsilon}+\tfrac{1}{2}\Big[\left\{A^{\rho}_{\lambda\upsilon}+\mathrm{B}^{\rho}_{\lambda\upsilon}\right\}_{\mu}-\left\{A^{\rho}_{\lambda\mu}+B^{\rho}_{\lambda\mu}\right\}_{\upsilon}$$
$$+A^{\rho}_{\alpha\mu}B^{\alpha}_{\lambda\upsilon}+B^{\rho}_{\alpha\mu}A^{\alpha}_{\lambda\upsilon}-A^{\rho}_{\varpi}B^{\alpha}_{\lambda\mu}-B^{\rho}_{\varpi}A^{\rho}_{\lambda\mu}\Big]$$

Now when (5.1) holds, we get from (5.2)

$$A^{\rho}_{\lambda\mu}={}^{a}S^{\rho}_{\lambda\mu}+U^{\rho}_{\lambda\mu},\quad B^{\rho}_{\lambda\mu}={}^{b}S^{\rho}_{\lambda\mu}+V^{\rho}_{\lambda\mu}$$

Putting these values in the above equation and contracting for ρ and ν, we get

$$ {}^{1}R_{\lambda\mu}(a,b)=H_{\lambda\mu}+\tfrac{1}{2}\Big[\left\{{}^{a}S_{\lambda}+{}^{b}S_{\lambda}+U_{\lambda}+V_{\lambda}\right\}_{\mu}-\left\{aS^{\alpha}_{\lambda\mu}+{}^{b}S^{\alpha}_{\lambda\mu}+U^{\alpha}_{\lambda\mu}+V^{\alpha}_{\lambda\mu}\right\}_{\alpha}$$
$$+\left({}^{a}S^{\alpha}_{\beta\mu}+U^{\alpha}_{\beta\mu}\right)\left({}^{b}S^{\beta}_{\lambda\alpha}+V^{\beta}_{\lambda o}\right)+\left({}^{b}S^{\alpha}_{\beta\mu}+V^{\beta}_{\alpha\mu}\right)\left({}^{a}S^{\beta}_{\lambda\alpha}+U^{\alpha}_{\lambda\alpha}\right)$$
$$-\left({}^{a}S_{\beta}+U_{\beta}\right)\left({}^{b}S^{\beta}_{\lambda\mu}+V^{\beta}_{\lambda\mu}\right)-\left({}^{b}S_{\beta}+V_{\beta}\right)\left({}^{a}S^{\beta}_{\lambda\mu}-U^{\beta}_{\lambda\mu}\right)\Big]$$

Interchanging λ and μ and subtracting, we obtain

$$2\ {}^{1}R_{[\lambda\mu]}(a,b)=\left\{{}^{a}S_{[\lambda}+{}^{b}S_{[\lambda}+U_{[\lambda}+V_{[\lambda}\right\}_{\mu]}-\left\{{}^{a}S^{\alpha}_{[\lambda\mu]}+{}^{b}S^{\alpha}_{[\lambda\mu]}+U^{\alpha}_{[\lambda\mu]}+V^{\alpha}_{[\lambda\mu]}\right\}$$
$$+\left({}^{a}S_{\beta[\mu}+U^{\alpha}_{\beta[\mu}\right)\left({}^{b}S^{\beta}_{\lambda]\alpha}+V^{\beta}_{\lambda]\alpha}\right)+\left({}^{b}S^{\beta}_{\beta[\mu}+V^{\alpha}_{\beta[\mu}\right)\left({}^{a}S^{\beta}_{\lambda]\alpha}+U^{\beta}_{\lambda]\alpha}\right)$$
$$-\left({}^{a}S_{\beta}+U_{\beta}\right)\left({}^{b}S^{\beta}_{[\lambda\mu]}+V^{\beta}_{[\lambda\mu]}\right)-\left({}^{b}S_{\beta}+V_{\beta}\right)\left({}^{a}S^{\beta}_{[\lambda\mu]}+U^{\beta}_{[\lambda\mu]}\right). \tag{5.12}$$

Hence we may state the following theorem :

Theorem 4. *If the equations (5.1) hold, then the equations (5.12) also hold, where* $U^{\rho}_{\lambda\mu}$, $V^{\rho}_{\lambda\mu}$ *are defined by (5.2a) and the Christoffel symbols with respect to which the covariant derivatives are taken are defined by (5.3), all expressed in terms of* $S^{\rho}_{\lambda\mu}$, ${}^{b}S^{\rho}_{\lambda\mu}$, $T^{\rho}_{\lambda\mu}$ *and the tensor* $g_{\lambda\mu}$.

In conclusion it may be stated that a generalised form of the field equations given by (1), (2) of § 1 may be given as follows :

$$g_{\underset{ab'}{\lambda\mu;\nu}}=0, \qquad \text{where} \qquad {}^{b'}\Gamma^{\lambda}_{\lambda\mu} = {}^{b}\Gamma^{\nu}_{\mu\lambda} \tag{5.13}$$

$$(i)\ \ {}^{a}S_{\lambda}+{}^{b}S_{\lambda}=0, \quad (ii)\ \ {}^{1}R_{(\lambda\mu)}(a,b)=0, \quad (iii)\ \ {}^{1}R_{[\lambda\mu]}{}'(a,b)=X_{[\lambda}(a,b)_{,\mu]}. \tag{5.14}$$

The equations (5.13) may be regarded as an algebraic system of n^3 equations to determine the $2n^3$ quantities $\left({}^{a}\Gamma^{\rho}_{\lambda\mu},\ {}^{b}\Gamma^{\rho}_{\lambda\mu}\right)$. If the system admits a solution and the solution is substituted in (5.14), we get a system of $n(n+1)$ differential equations to determine the same number of quantities $g_{\lambda\mu}$, $X_{\lambda}(a,b)$. On the other hand one could start with the n^2 differential equations (ii), (iii) of (5.14) with respect to the unknowns $\left({}^{a}\Gamma^{\rho}_{\lambda\mu},\ {}^{b}\Gamma^{\rho}_{\lambda\mu}\right)$ which unknowns are subject to the algebraic system (i). The system (5.13) is then a system of differential equations with respect to the unknowns $g_{\lambda\mu}$. However, if we take $a=b$, the equations (5.13), (5.14) reduce to the equations (1), (2) of § 1, and all the theorems stated in this section become equivalent to the corresponding theorems obtained by Hlavatý (Hlavatý, 1957).

In any case, it is believed that the concepts of generalised covariant derivative and generalised curvature tensor given in this paper may lead to some fruitful results.

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STEADY HYDROMAGNETIC FLOW IN A CHANNEL WITH POROUS AND CONDUCTING WALLS IN THE PRESENCE OF A TRANSVERSE MAGNETIC FIELD.

By

M. Ray and J. C. Agrawal, *Agra*

(*Received—February 8, 1964*)

An exact solution of modified Navier-Stokes equations governing the steady flow of a conducting fluid between two parallel conducting porous infinite plane walls is given. When the suction or injection at the walls is made to vanish the flow reduces to the one described by Chang and Yen

Introduction. Hartmann and Lazarus (1937) were the first to experiment on the hydromagnetic flow of a liquid conductor, through ducts of rectangular and circular cross-section. The theory of flow between parallel non-conducting walls has been reproduced at many places (e.g. Cowling, 1957). If the magnetic field H_0 is perpendicular to the non-conducting walls having equations ($z = \pm L$) and if P be the pressure gradient and M the Hartmann number $[M = \mu H_0 L(\sigma/\rho_\nu)^{\frac{1}{2}}]$, the velocity is given by

$$v = \frac{PL^2}{\rho\nu M}\left[\frac{\cosh M - \cosh\left(\frac{Mz}{L}\right)}{\sinh M}\right]$$

which shows that if M be large, v is practically constant except in a narrow region—a layer of width $\sim L/M$ at each wall, the velocity away from the walls being approximately $\frac{\rho L}{H_0(\sigma\rho\nu)^{\frac{1}{2}}}$, i.e. proportional to L and inversely proportional to H_0. The electric current density in the central part of the channel is uniform, there is a return current in the boundary layers, which balances the central currents. The magnetic lines of force can be supposed to move in the channel with velocity w such that in the central part of the channel P is just balanced by the drag $-\sigma H_0^2(v-w)$, (because of a polarization electric field). Thus $v-w = \frac{\rho}{\sigma H_0^2} \simeq \frac{v}{M}$. Thus the lines of force in the fluid move with a velocity $v\left(1-\frac{1}{M}\right)$ only a little smaller than v, this reduces the drag. If the polarization is cut down (e.g. because of the conductivity of the walls) the drag becomes important, and v is correspondingly reduced (Chang, 1961 and Shercliff, 1956).

The theory of flow through ducts of rectangular and circular cross section was considered by Shercliff (1953, 1955, 1956a,b,c) Uflyand (1960) and Chang and Lundgren (1961). Shercliff (1953) gave an exact solution for the rectangular case with non-conducting walls, the magnetic field H_0 being perpendicular to the two walls. Chang and Lundgren (1961) considered the effect of perfectly conducting walls on the flow in rectangular and circular ducts. For large M they have shown that except in the boundary layer of thickness of the order of $\frac{1}{M}$, $v \sim \frac{1}{M} \frac{\phi+1}{M\phi+1}$ where $\phi = 0$ for non-conducting walls and $\phi = \infty$ for perfectly conducting walls. Thus $v \sim \frac{1}{M}$ for non-conducting walls and $v \sim \frac{1}{M^2}$ for perfectly conducting walls, increasing the wall conductivity decreases the velocity.

The stationary flow of a conducting fluid in channels with non-conducting, porous walls has recently been decribed by Agrawal and Agrawal (1963) and Agrawal (1963).

The object of the present paper is to consider the effect of wall-conductance on the $M-H-D$ channel flow when channel walls are porous and provide exact solutions, for this problem. Numerical results and detailed discussions have not been reproduced to avoid the increase in the length of the paper.

The Governing Equations. The fundamental equations of magneto-hydrodynamics are the simultaneous equations of modified Navier-Stokes and Maxwell's equations taking account of the interaction between the field and motion. Ignoring Maxwell's displacement currents and accumulation of charges, and assuming Lorentz force to be the only body force in the field, equations of the hydromagnetic flow are : (Cowling, 1957).

$$\text{Curl } \boldsymbol{H} = 4\pi \boldsymbol{j} \tag{1}$$

$$\text{Curl } \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \tag{2}$$

$$\text{div } \boldsymbol{H} = 0 \tag{3}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{E}+\mu \boldsymbol{v} \times \boldsymbol{H}) \tag{4}$$

$$\text{div } \boldsymbol{v} = 0 \tag{5}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho} \nabla p + \nu\nabla^2 \boldsymbol{v} + \frac{\mu}{\rho} \boldsymbol{j} \times \boldsymbol{H} \tag{6}$$

where $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{E}$ the electric field, $\boldsymbol{v}$ the velocity, $\boldsymbol{j}$ the current density, μ the permeability, ρ the density, ν the kinematic coefficient of viscosity and p the pressure and c.g.s. electromagnetic units have been employed.

Eliminating $\boldsymbol{E}$ and $\boldsymbol{j}$ in (1), (2), (4) and $\boldsymbol{j}$ in (1) and (6) making use of (3) and (5) we get,

$$\frac{\partial \boldsymbol{H}}{\partial t} = \text{curl}\,(\boldsymbol{v} \times \boldsymbol{H}) + \lambda \nabla^2 \boldsymbol{H} \tag{7}$$

where

$$\lambda = \left(\frac{1}{4\pi\mu\sigma} \right) \tag{8}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla \left(\frac{p}{\rho} + \frac{\mu}{8\pi\rho} |\boldsymbol{H}|^2 \right) + \frac{\mu}{4\pi\rho} (\boldsymbol{H}\cdot\nabla)\boldsymbol{H} + \nu\nabla^2 \boldsymbol{v} \tag{9}$$

We select a righthanded Cartesian system such that the z axis is perpendicular to the walls and the walls have the equations $z = \pm L$. Walls have been assumed to be equally electrically conducting, of equal thickness h such that $h/L << 1$ and to have the same permeability as that of the fluid. The fluid flows between these walls in the direction parallel to the x axis and a uniform magnetic field H_0 is applied in the direction of z axis. The equations governing the flow are simplified on account of the following assumptions :

(i) The flow is steady and laminar and hence $\frac{\partial}{\partial t} = 0$

(ii) Velocity, magnetic field and the pressure are independent of the space function y and therefore

$$\frac{\partial \boldsymbol{v}}{\partial y} = \frac{\partial \boldsymbol{H}}{\partial y} = \frac{\partial p}{\partial y} = 0.$$

(iii) The fluid injection rate at one wall is equal to the fluid suction rate at the other. Thus

$$v_z(L) = v_z(-L) = v_z(z) = v_0 = \text{constant}, \tag{10}$$

where $v_2(z)$ is the cross flow velocity (parallel to Oz) at any point (x, y, z) in the channel.

(iv) The component of velocity parallel to the walls is taken in the direction of x axis so that $v_y = 0$ and equation of continuity (5) then reduces to

$$\frac{\partial v_x}{\partial x} = 0 \tag{11}$$

where v_x denotes the longitudinal component of velocity. Thus v_x is independent of x and on account of assumption (ii) v_x is a function of z only.

(v) The externally applied magnetic field H_0 parallel to Oz fixes the normal component of the magnetic field at the surfaces of the planes $z = \pm L$ and that this is the only field impressed. There is no applied electric field and also there are no free charges, and hence $\boldsymbol{E} = 0$.

Since the liquid near the meridian plane is moving faster than that near the walls, it tends to pull out the lines of force in the direction of motion and the field thus acquires a component H_x parallel to Ox.

Equation (3) reduces to

$$\frac{\partial H_x}{\partial x} = 0 \tag{12}$$

which again shows that H_x is a function of z only.

Now we have :

$$\boldsymbol{v} = (v_x, 0, v_0)$$

and

$$\boldsymbol{H} = (H_x, 0, H_0),$$

where quantities in the parentheses denote respectively the x, y, z components of the corresponding vectors.

Thus,

$$\boldsymbol{v} \times \boldsymbol{H} = (0, v_0 H_x - v_x H_0, 0) \tag{13}$$

Since $\boldsymbol{E} = 0$, components of equation (4) are :

$$j_x = 0 \tag{14}$$

$$j_y = \mu\sigma(v_0 H_x - v_x H_0) \tag{15}$$

$$j_z = 0 \tag{16}$$

Equation (2) has only y component, which gives

$$\frac{\partial H_x}{\partial z} = 4\pi j_y \tag{17}$$

Equations (15) and (17) give

$$\frac{\partial H_x}{\partial z} = 4\pi j_y = \frac{1}{\lambda}(v_0 H_x - v_x H_0) \tag{18}$$

With the above assumptions and results obtained, the equations (7) and (9) reduce to the following three equations :

$$H_0 \frac{dv_x}{dz} - v_0 \frac{dH_x}{dz} + \lambda \frac{d^2 H_x}{dz^2} = 0 \tag{19}$$

$$\frac{\partial p}{\partial z} + \frac{\mu}{8\pi} \frac{dH_x^2}{dz} = 0 \tag{20}$$

and

$$v_0 \frac{dv_x}{dz} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{d^2 v_x}{dz^2} + \frac{\mu H_0}{4\pi\rho} \frac{dH_x}{dz} \tag{21}$$

Differentiating (20) with respect to x, $\frac{\partial^2 p}{\partial x \partial z} = 0 = \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right)$ showing that $\frac{\partial p}{\partial x}$ is independent of z. Differentiating (21) with respect to x we have $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = 0$, which shows that $\frac{\partial p}{\partial x}$ is independent of x also. We, therefore, write :

$$-\frac{\partial p}{\partial x} = P = \text{constant.} \tag{22}$$

Equations (20) and (22) give :

$$p + Px + \frac{\mu}{8\pi} H_x^2 = \text{constant.} \tag{23}$$

After determining H_x, the pressure p can be determined.

Non-Dimensional form of the Equations. Equations (19) and (21) are the simultaneous differential equations for v_x and H_x and (23) gives the variation of pressure. We define the following non-dimensional quantities :

$$\xi = (z/L),\ \Pi = \frac{p}{PL},\ H = \frac{H_x}{H_0},\ v = v_x\sqrt{\frac{\rho}{PL}},\ \xi = \frac{x}{L} \tag{24}$$

R = Reynolds number for the longitudinal flow

$$= \sqrt{\frac{PL}{\rho}} \cdot \frac{L}{\nu} \tag{25}$$

R_m = Magnetic Reynolds number

$$= \sqrt{\frac{PL}{\rho}} \cdot \frac{L}{\lambda} \tag{26}$$

$$S^2 = (\text{Magnetic pressure number})^2 = \frac{\mu H_0^2}{4\pi PL} \tag{27}$$

$$= \frac{M^2}{RR_m} \text{ where } M \text{ is the Hartmann number}$$

and

R_c = Reynolds number for the cross-flow

$$= \sqrt{\frac{PL}{\rho}} \cdot \frac{1}{v_0} \tag{28}$$

With these substitutions equations (19), (21) and (23) reduce to :

$$\frac{dv}{d\zeta} - \frac{1}{R_c}\frac{dH}{d\zeta} + \frac{1}{R_m}\frac{d^2H}{d\zeta^2} = 0 \tag{29}$$

$$1+S^2\frac{dH}{d\zeta} - \frac{1}{R_c}\frac{dv}{d\zeta} + \frac{1}{R}\frac{d^2v}{d\zeta^2} = 0 \tag{30}$$

and

$$\Pi + \xi + \tfrac{1}{2}H^2 = \text{constant}. \tag{31}$$

Boundary Conditions. (a) No slip conditions at the channel walls $z = \pm L$ are represented by

and

$$\left.\begin{aligned} \zeta &= 1, & v &= 0 \\ \zeta &= -1 & v &= 0. \end{aligned}\right\} \tag{32}$$

(b) The whole space can be classified broadly into three regions

(i) the fluid region,

(ii) the channel walls each of constant thickness h and electrical conductivity σ_w and

(iii) outside of the channel walls.

A uniform magnetic field H_0 acts in the Oz direction. The two boundaries $z = L$ and $z = -L$ are the interface of regions (i) and (ii) and the two boundaries $z = (L+h)$ and $z = -(L+h)$ are the interface of regions (ii) and (iii). Since we are interested in the physics inside the channel we need the bundary conditions on H at the interface of (i) and (ii) i.e the boundaries $\zeta = \pm 1$. Chang and Lundgren (1961) have investigated these boundary conditions in case of a parallel sided duct to be

$$\phi\,\frac{dH}{d\zeta} + H = 0 \text{ at } \zeta = 1$$

and

$$\phi \frac{dH}{d\zeta} - H = 0 \text{ at } \zeta = -1 \tag{33}$$

(Chang and Yen, 1962).

where

$$\phi = \frac{\sigma_w h}{\sigma L}.$$

The electrical conductance ratio ϕ may b different for the two walls.

If the thicknesses and conductivities of the upper and lower walls be respectively h_1 and h_2 and σw_1 and σw_2, then the boundary conditions will take the form

$$\phi_1 \frac{dH}{d\zeta} + H = 0 \text{ at } \zeta = 1$$

and

$$\phi_2 \frac{dH}{d\zeta} - H = 0 \text{ at } \zeta = -1$$

where

$$\phi_1 = \frac{\sigma w_1 h_1}{\sigma L} \text{ and } \phi_2 = \frac{\sigma w_2 h_2}{\sigma L}$$

and these are more general boundary conditions than (33). However, we have taken boundary conditions (33) for simplicity of results. It may also be noted that the cases of non-conducting walls $\sigma_w = 0$(Agrawal and Agrawal, 1963) and perfectly conducting walls $\sigma_w = \infty$ are included in the boundary condition (33) with $\phi \to 0$ and $\phi \to \infty$ respectively.

Solution of Equations. The first integrals of equations (29) and (30) are respectively

$$\frac{1}{R_m} \frac{dH}{d\zeta} - \frac{1}{R_c} H + v = C_1 \tag{34}$$

and

$$\frac{1}{R} \frac{dv}{d\zeta} - \frac{1}{R_c} v + S^2 H + \zeta = C_2 \tag{35}$$

Eliminating H in these two equations we get :

$$\frac{d^2 v}{d\zeta^2} - \left(\frac{R+R_m}{R_c} \right) \frac{dv}{d\zeta} + \left(\frac{RR_m}{R_c^2} - M^2 \right) v$$

$$= \frac{RR_m}{R_c} \zeta - R - M^2 C_1 - \frac{RR_m}{R_c} C_2 \tag{36}$$

Solving the equation (36) by usual procedure and feeding the solution in (35) to get H and taking account of the boundary conditions (32) and (33), the solutions are :

$$v = C_3(e^{\alpha} - e^{\alpha\zeta}) + C_4(e^{\beta} - e^{\beta\zeta}) + \frac{RR_m}{R} \frac{(1-\zeta)}{\alpha\beta} \tag{37}$$

and

$$S^2 H = \left(\frac{1}{R_c} - \frac{\alpha}{R} \right) (\alpha\phi \sinh \alpha + \cosh \alpha - e^{\alpha\zeta}) C_3$$

$$+ \left(\frac{1}{R_c} - \frac{\beta}{R} \right) (\beta\phi \sinh \beta + \cosh \beta - e^{\beta\zeta}) C_4 + \left(1 - \frac{RR_m}{R_c^2 \alpha\beta} \right) \zeta \tag{38}$$

where α, β are the roots of the quadratic equation

$$\eta^2 - \left(\frac{R+R_m}{R_c} \right) \eta + \left(\frac{RR_m}{R^2} - M^2 \right) = 0, \tag{39}$$

$$C_3 = -\frac{\dfrac{R\,R_m}{R_c^2}\,\dfrac{\phi}{\alpha\beta}\,(\beta \coth \beta - 1) - \dfrac{R_m}{R_c}\,\dfrac{1}{\alpha}\,(\beta\,\phi \coth \beta + 1) - (\phi + 1)}{\sinh \alpha \left[\dfrac{\phi}{R_c}\,(\alpha \coth \alpha - \beta \coth \beta) - \dfrac{\phi}{R}\,(\alpha^2 \coth \alpha - \beta^2 \coth \beta) - \dfrac{1}{R}\,(\alpha - \beta) \right]} \tag{40}$$

and

$$C_4 = \frac{\dfrac{R\,R_m}{R_c^2}\,\dfrac{\phi}{\alpha\beta}\,(\alpha \coth \alpha - 1) - \dfrac{R_m}{R_c}\,\dfrac{1}{\beta}\,(\alpha\phi \coth \alpha + 1) - (\phi + 1)}{\sinh \beta \left[\dfrac{\phi}{R_c}\,(\alpha \coth \alpha - \beta \coth \beta) - \dfrac{\phi}{R}\,(\alpha^2 \coth \alpha - \beta^2 \coth \beta) - \dfrac{1}{R}\,(\alpha - \beta) \right]} \tag{41}$$

Pressure variation may be obtained from equation (31) after substituting the value of H from (38).

This completes the solution of the hydromagnetic equations. It may be noted that the results apply equally well for either direction of cross-flow (due to fluid suction at one wall and injection at the other). Positive value of R_c corresponds to injection at the wall $z = -L$ while the negative value of R_c corresponds to injection at the wall $z = L$.

Limiting Forms of The Flow

(I) When the walls of the channel are non-conducting, there is a change in the boundary conditions for the magnetic field which is obtained by putting $\phi = 0$. For the case $\phi = 0$, results agree with those given by Agrawal and Agrawal (1963).

(II) If the cross-flow be stopped i.e. $v_0 = 0$ and $1/R_c = 0$, the results reduce to those obtained by Chang and Yen (1962).

(III) Results in the case of perfectly conducting walls can be derived from (37) and (38) by making $\phi \to \infty$.

Only the values C_3 and C_4 are changed.
In this case :

$$C_3 = -\frac{\dfrac{R\,R_m}{R_c^2}\,\dfrac{1}{\alpha\beta}\,(\beta \coth \beta - 1) - \dfrac{R_m}{R_c}\,\dfrac{\beta}{\alpha}\,\coth \beta - 1}{\sinh \alpha \left[\dfrac{\alpha \coth \alpha - \beta \coth \beta}{R_c} - \dfrac{\alpha^2 \coth \alpha - \beta^2 \coth \beta}{R} \right]}$$

and

$$C_4 = \frac{(R\,R_m/R_c^2\alpha\beta)(\alpha \coth a - 1) - (\alpha\,R_m/\beta R_c) \coth \alpha - 1}{\sinh \beta \left[\dfrac{\alpha \coth \alpha - \beta \coth \beta}{R_c} - \dfrac{\alpha^2 \coth \alpha - \beta^2 \coth \beta}{R} \right]}$$

(IV) In the limiting case (I) if the cross flow be stopped, i.e. $v_0 = 0$ and hence $1/R_c = 0$ then $\alpha = -\beta = M$. The results in this case agree with those of Hartmann flow when $\boldsymbol{E}$ applied electric field is set equal to zero.

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HYDROMAGNETIC FLOW OVER A PERFECTLY CONDUCTING PLANE WALL WITH UNIFORM SUCTION

By

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Introduction. In the present paper is studied the problem of steady flow of an electrically conducting, incompressible, viscous fluid of constant electric and magnetic properties over a perfectly conducting plane wall with uniform suction in the presence of a transverse magnetic field. Gupta (1960) has treated the problem of flow past a non-conducting porous plane wall (for both cases of suction and injection). Due to some inconsistencies in his paper the problem was taken again by Kakutani (1961) who has shown that no steady solution can be obtained for the case of injection. However, for the case of uniform suction he has obtained an exact solution for the steady state when the magnetic pressure number is smaller than unity. The solution to the problem of the hydromagnetic flow over a perfectly conducting infinite plane wall with uniform suction in the presence of a transverse magnetic field does not appear to be given anywhere. The paper has been written with the object of providing an exact solution of this problem.

The Governing Equations. The basic equations expressing the interactions between the fluid motion and the magnetic field are contained in Maxwell's equations and in the equations of hydrodynamics suitably modified. Since in this problem we are not concerned with the effects which may be related in any way to the propagation of electromagnetic waves, we are justified in ignoring displacement currents in the Maxwell's equations. Variations of the charge density affect the equation of conservation of charge by terms of the order v^2/c^2 and terms of this order we have ignored. If H is the magnetic field, E the electric intensity, j the current density, μ the permeability, ρ the density, ν the Kinematic coefficient of viscosity and p the pressure and c.g.s. electromagnetic units are employed, assuming that the only body force in the field is the Lorentz-force, the equations governing the flow of an electrically-conducting incompressible, viscous fluid in the presence of a magnetic field are (Cowling, 1957)

$$\nabla \times H = 4\pi j \tag{1}$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \tag{2}$$

$$\nabla \cdot H = 0 \tag{3}$$

$$j = \sigma(E + \mu V \times H) \tag{4}$$

$$\nabla \cdot V = 0 \tag{5}$$

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla p + \nu \nabla^2 V + \frac{\mu}{\rho}\, j \times H \tag{6}$$

Eliminating E and j from (1), (2) and (4) and between (1) and (6) we have respectively

$$\frac{\partial H}{\partial t} = \nabla\times(V\times H)+\lambda\nabla^2 H \tag{7}$$

where $\lambda = (1/4\pi\mu\sigma)$ and

$$\frac{\partial V}{\partial t}+(V\cdot\nabla)V = -\nabla\left(\frac{p}{\rho}+\frac{\mu}{8\pi\rho}\,|H|^2\right)+\frac{\mu}{4\pi\rho}(H\cdot\nabla)H+\nu\nabla^2 V. \tag{8}$$

Now we take a right handed system of coordinate axes with origin O at an arbitrary point on the wall which is assumed to be perfectly conducting and to have the same permeability as that of the fluid, and use the Cartesian coordinates (x, y, z) with axes Ox and Oz along and perpendicular to the wall respectively so that equation of the plane is $z = 0$. Equations governing the flow described above are simplified on account of the following assumptions :

(I) All physical quantities are supposed to be the functions of space coordinate z alone.

(II) Velocity field is assumed to be two dimensional i.e. $V = (v_x, 0, v_z)$ where the quantities in the parentheses denote respectively the x, y, z components of the corresponding vector.

Equation (5) on the above assumptions leads to

$$\frac{\partial v_z}{\partial z} = 0 \quad \text{or} \quad v_z = v_0 = \text{constant} \tag{9}$$

(III) A uniform magnetic field is applied parallel to Oz. Since the liquid away from the plane $z = 0$ moves faster than that near the wall it tends to pull out the lines of force in its direction of motion. Thus the field acquires a component H_x parallel to the motion.

Thus
$$H = (H_x, 0, H_0)$$

(IV) The flow is steady and laminar hence $(\partial/\partial t) = 0$. Now

$$V = (v_x, 0, v_0)$$

$$H = (H_x, 0, H_0)$$

and hence

$$V\times H = (0, v_0H_x - v_xH_0, 0) \tag{10}$$

Since there is no applied electric field and hence charges, equation (4) for the current-density gives the components of the current :

$$j_x = 0. \tag{11}$$

$$j_y = \mu\sigma(v_0H_x - v_xH_0). \tag{12}$$

$$j_z = 0. \tag{13}$$

Obviously, in this problem, curl H, $V\times H$ and j have only y component.

Equation (1) gives

$$(\text{Curl } H)_y = \frac{\partial H_x}{\partial z} = 4\pi j_y \tag{14}$$

and equation (12) with (14) gives

$$\frac{\partial H_x}{\partial z} = 4\pi j_y = \frac{1}{\lambda}(v_0 H_x - v_x H_0) \tag{15}$$

With these assumptions and the consequences of assumptions stated above, the equations (7) and (8) reduce to the following three equations :

$$H_0 \frac{dv_x}{dz} - v_0 \frac{dH_x}{dz} + \lambda \frac{d^2 H_x}{dz^2} = 0 \tag{16}$$

$$\frac{dp}{dz} + \frac{\mu}{8\pi} \frac{dH_x^2}{dz} = 0 \tag{17}$$

and

$$v_0 \frac{dv_x}{dz} = \nu \frac{d^2 v_x}{dz^2} + \frac{\mu H_0}{4\pi\rho} \frac{dH_x}{dz} \tag{18}$$

The boundary conditions for the velocity are :

No slip condition at the wall, given by

$$z = 0 : v_x = 0 \tag{19}$$

The component of velocity at far distance from the wall tends to a uniform value U.

$$z \to \infty : v_x \to U \tag{20}$$

The boundary conditions for the magnetic field are :

Since the wall has been assumed to be perfectly conducting (conductivity of the wall $= \infty$), and its permeability same as that of the fluid (Chang and Lundren 1961)

$$z = 0 : \frac{dH_x}{dz} = 0 \tag{21}$$

Since the x component of velocity, at far distance from the wall tends to a uniform value U, the x component of magnetic field tends to 0.

$$z \to \infty : \quad H_x \to 0 \tag{22}$$

Since the velocity and magnetic field both tend to uniform values at far distance from the wall, their derivatives with respect to z tend to zero.

$$z \to \infty : \frac{dv_x}{dz} \to 0 \text{ and } \frac{dH_x}{dz} \to 0. \tag{23}$$

Solution of Equations Equation (17) on integration, taking account of boundary condition (22) gives

$$p + \frac{\mu}{8\pi} H_x^2 = \Pi \tag{24}$$

where Π is the pressure at infinity.

The first integrals of equations (16) and (18) making use of boundary conditions at infinity are respectively

$$H_0 v_x - v_0 H_x + \lambda \frac{dH_x}{dz} = H_0 U \tag{25}$$

and
$$v_0 v_x = \nu \frac{dv_x}{dz} + \frac{\mu H_0}{4\pi\rho} H_x + v_0 U \tag{26}$$

Elimination of H_x in these two equations gives

$$\frac{d^2v_x}{dz^2} - \frac{v_0}{\nu}(1+\rho_m)\frac{dv_x}{dz} + \frac{v_0^2}{\nu\lambda}(1-S)v_x = \frac{v_0^2}{\nu\lambda}(1-S)U \tag{27}$$

where

$$\rho_m = \text{the magnetic Prandtle number} = (\nu/\lambda) \tag{28}$$

$$S = \text{the magnetic pressure number} = \frac{\mu H_0^2}{4\pi\rho v_0^2} \tag{29}$$

Solving the equation (27) by usual process we get

$$v_x = U + Ae^{\alpha z} + Be^{\beta z} \tag{30}$$

where α and β are the roots of the quadratic

$$\xi^2 - \frac{v_0}{\nu}(1+\rho_m)\xi + \left(\frac{v_0^2}{\nu\lambda}\right)(1-S) = 0$$

or

$$\alpha, \beta = \frac{v_0}{2\nu}\left[(1+\rho_m) \pm \sqrt{(1-\rho_m)^2 + 4S\rho_m}\right] \tag{31}$$

and constants of integration A and B are to be determined from the boundary conditions on the wall. We discuss the solution (30) first.

Positive value of v_0 represents injection at the wall whereas negative value of v_0 corresponds to suction. In both these cases the sign of S depends on μ and not on v_0.

Firstly we take $v_0 > 1$ which corresponds to uniform injection. In this case both α and β are positive when $S < 1$ and v_x given by (30) diverges at infinity. When $S = 1$, one of α and β is positive the other being zero and again v_x diverges at infinity. When $S > 1$, one of α, β or say α is positive and β negative. From the boundary condition (20) we see that $A = 0$; B can be determined from the boundary condition (19) alone and there remains no freedom to satisfy the boundary condition (21). Thus for all values of S in the case of injection, no steady solution is obtained.

Now we take $v_0 < 1$ which corresponds to uniform suction. When $S \geqslant 1$ we again see that no steady solution is obtained for similar reasons as in the preceding case. When $S < 1$, both α and β are negative, so that the boundary condition at infinity for U is satisfied for any values of A and B. Equations (26) and (30) give,

$$H_x = \frac{4\pi\rho}{\mu H_0}\left[(v_0 - \nu\alpha)Ae^{\alpha z} + (v_0 - \nu\beta)Be^{\beta z}\right] \tag{32}$$

and

$$\frac{dH_x}{dz} = \frac{4\pi\rho}{\mu H_0}\left[(v_0 - \nu\alpha)\alpha Ae^{\alpha z} + (v_0 - \nu\beta)Be^{\beta z}\right] \tag{33}$$

Equations (30), (32) and (33) already satisfy the boundary conditions (20), (22) and (23) respectively. Boundary conditions (19) on equation (30) and (21) on equation (33) give respectively

$$A+B+U=0 \tag{34}$$

and

$$\alpha(v_0-\nu\alpha)A+\beta(v_0-\nu\beta)B=0 \tag{35}$$

Solving (A) and (B)

$$A=-\frac{(1+\rho_m-\gamma)(1-\rho_m+\gamma)}{4\gamma\rho_m}U \tag{36}$$

and

$$B=\frac{(1+\rho_m+\gamma)(1-\rho_m-\gamma)}{4\gamma\rho_m}U \tag{37}$$

where we write γ for $\sqrt{(1-\rho_m)^2+4S\rho_m}$

Writing the values of A and B given by (36) and (37) in (30) and (32) we get

$$\frac{v_x}{U}=1-\frac{1}{4\gamma\rho_m}\left[(1+\rho_m-\gamma)(1-\rho_m+\gamma)e^{\alpha z}-(1+\rho_m+\gamma)(1-\rho_m-\gamma)e^{\beta z}\right] \tag{38}$$

and

$$\frac{H_x}{H_0}=\frac{U}{2\gamma v_0}\left[(1+\rho_m-\gamma)e^{\alpha z}-(1+\rho_m+\gamma)e^{\beta z}\right] \tag{39}$$

Equations (24), (38), (39) constitute the exact solutions for the magnetohydrodynamic flow over a perfectly conducting plane wall with uniform suction when a uniform transverse magnetic field is impressed.

The electric-current density has only one component which can be obtained by substituting v_x and H_x from (38) and (39) in (12).

Since suction checks the growth of the boundary layer thickness, the asymptotic steady state may be obtained . But the magnetic field counteracts this checking of boundary layer thickness by suction and as $S\to 1$ magnetic field balances with the checking effect of suction. Hence no steady solution can be obtained for $S \geqslant 1$.

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CALCULATION OF ONE-ELECTRON LEVELS IN A DISORDERED LATTICE BY THE MOMENTUM-TRACE METHOD

BY

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Introduction. The method of moments has been used for determining the frequency spectrum of the vibrations of an ordered lattice (Montroll, 1942) and of lattices consisting of isotope mixtures of different mass ratios and concentrations (Domb, Maradudin, Montroll, Weiss, 1959). A similar method can be used to determine the electronic energy spectrum in liquids. However, it is important to remember that the number of modes of vibration in a finite solid is finite, whereas the electronic energy spectrum extends to infinity. The moments of this matrix are infinite, i.e. they do not exist. It is therefore necessary to consider energies only to some finite upper limit. This is formally achieved by replacing the Hamiltonian H by HP (to be denoted by W) where P is a projection operator. An approximation to the spectrum of H in the neighbourhood of the wave number k can be obtained by taking a suitable P such that HP is an $\nu \times \nu$ matrix containing all those elements $H_{k,k'}$ which are not negligible. By taking a larger ν the spectrum of HP can be made to approximate more closely at least in principle to the spectrum of H obtained by the exact diagonalization of the infinite energy matrix.

The Momentum-Trace Method. Let λ_i be the eigenvalues of an $m \times m$ matrix A; then λ_i^r are the eigenvalues of A^r.

$$\text{Trace} \quad A = \Sigma\, \lambda_i$$

$$\text{Trace} \quad A^r = \Sigma\, \lambda_i^r \quad r = 1, 2, 3, \ldots \tag{1}$$

If λ's are so closely spaced that a distribution function $D(\lambda)$ can be defined which denotes the fraction of eigenvalues in the range $(\lambda, \lambda+d\lambda)$ then

$$\frac{\text{Trace } A^r}{m} = \int_{\lambda_L}^{\lambda_G} \lambda^r\, D(\lambda) d\lambda \tag{2}$$

where λ_L is the lowest and λ_G is the greatest eigenvalue of A. But the average of the r^{th} power of the eigenvalues is defined to be the r^{th} moment. Therefore

$$\mu_r = \int_{\lambda_L}^{\lambda_G} \lambda^r\, D(\lambda) d\lambda \tag{3}$$

Montroll and others have given in the above mentioned papers a technique to calculate the various moments of the matrix whose eigenvalues give the vibration frequencies and then the distribution function $D(\lambda)$ is calculated by expanding it in Legendre Polynomials or power series.

In the following, a method for calculating the moments of HP is given. H is the energy matrix for an electron in a one dimensional disordered lattice representing a liquid.

The Model. The model consists of a lattice of length L of N atoms randomly distributed, such that the interatomic distances are statistically independent and have the same probability distribution $s(r)$. The Fourier transform of $s(r)$ is $f(k)$. $x_1, x_2, \ldots, x_N$ denote the positions of the atoms and $r_1, r_2, \ldots, r_N$ the interatomic distances.

$$x_1 = r_1, \quad x_2 = r_1+r_2, \ldots x_N = r_1+r_2+\ldots+r_N \tag{4}$$

The potential energy $U(x)$ of an electron is

$$U(x) = \sum_{i=1}^{N} u(x-x_i) \tag{5}$$

where u is its potential energy due to one atom. In weak binding approximation the unperturbed wave functions are

$$\frac{1}{\sqrt{L}}\, e^{ilx}$$

When periodic boundary conditions are applied then

$$k = \frac{2n\pi}{L} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{6}$$

The potential energy is treated as perturbation. The matrix elements $u_{k,k'}$ of the perturbation are

$$U_{k,k'} = \frac{1}{L}\int_0^L e^{-ikx}U(x)e^{ik'x}dx$$

$$= \frac{1}{L}\, v(k'-k)\rho(k'-k) \qquad k \neq k' \tag{7}$$

$$U_{k,k} = 0$$

where $v(k'-k) = \int u\, e^{i(k'-k)x}\, dx$ (8)

and ρ is the structure factor :

$$\rho(k'-k) = \sum_{i=1}^{N} \exp\{i(k'-k)x_i\}$$

$$= \sum_{i=1}^{N} \exp\{i(k'-k)\sum_{j=1}^{i} r_j\} \tag{9}$$

Thus the matrix elements are not fixed but are functions of $r_1, r_2, \ldots r_N$. Hence the moments are functions of $r_1, r_2, \ldots, r_N$. These moments are statistically averaged with respect to the distribution function $F(r_1, r_2, \ldots, r_N) = s(r_1)\, s(r_2) \ldots s(r_N)$. The mean moment provides the mean distribution function $D(E)$.

Statistical Averages of the Products of Phase Sums. To determine the average of the moments the averages of the products of the phase sums are needed. If

$$\rho(k) = e^{ikr_1} + e^{ik(r_1+r_2)} + \ldots + e^{ik(r_1+r_2+\ldots+r_N\cdot)}$$

then in the limit $N \to \infty$

$$<\rho(k_1)\,\rho(k_2) \ldots \rho(k_n)> = \sum_{K_1} g(K_1) + \sum_{K_1, K_2} g(K_1)\, g(K_2) + \ldots + \sum_{K_1, \ldots, K_n} g(K_1)\, g(K_2) \ldots g(K_n) \tag{10}$$

where $K_1 = k_1+k_2+\ldots+k_n$ and in the j-th sum $K_1, K_2, \ldots, K_j$ are distinct partial sums of $k_1, k_2, \ldots, k_n$ such that $K_h \subset K_{h-1}$. If the numerical values of $k_1, k_2, \ldots, k_n$ are such that a certain $K_i = 0$ then

$$g(K_i) = N$$

otherwise,

$$g(K_i) = f(K_i)[1-f(K_i)]^{-1}$$

This is a generalization of the result established earlier (Sah and Eisenschitz, 1960). For example, if $k_1 \neq -k_2$, $k_1 \neq 0$, $k_2 \neq 0$

$$<\rho(k_1)\,\rho(k_2)> = \frac{f(k_1+k_2)}{1-f(k_1+k_2)} \cdot \left[1 + \frac{f(k_1)}{1-f(k_1)} + \frac{f(k_2)}{1-f(k_2)}\right] \tag{11}$$

and if $k_1 \neq 0$

$$<\rho(k_1)\,\rho(-k_1)> = N\left[1 + \frac{f(k_1)}{1-f(k_1)} + \frac{f(-k_1)}{1-f(-k_1)}\right] \tag{12}$$

Similarly, if $k_1+k_2+k_3 \neq 0, \quad k_i+k_j \neq 0, \quad k_i \neq 0$

$$<\rho(k_1)\,\rho(k_2)\,\rho(k_3)> = \frac{f(k_1+k_2+k_3)}{1-f(k_1+k_2+k_3)}\left[1 + \sum_{i=1}^{3} \frac{f(k_i)}{1-f(k_i)}\right.$$

$$\left. + \sum_{i,j=1}^{3}{}' \frac{f(k_i+k_j)}{1-f(k_i+k_j)} \left\{1 + \frac{f(k_i)}{1-f(k_i)} + \frac{f(k_j)}{1-f(k_j)}\right\}\right] \tag{13}$$

Calculation of Moments. Let k vary from k_1 to k_2 in $HP = W$ and let

$$\frac{L}{2\pi}(k_1-k_2) = \nu \text{ (say)}$$

Then

$$\mu_0 = \frac{1}{\nu}\sum_{k=k_1}^{k_2} \tag{14}$$

$$\mu_1 = \frac{1}{\nu}\sum_{k=k_1}^{k_2} W_{kk} \tag{15}$$

$$\mu_2 = \frac{1}{\nu}\sum_{k,\, k'} W_{kk'} W_{k'k} \tag{16}$$

....

$$\mu_n = \frac{1}{\nu}\sum_{k,\, k', \ldots k^{(n-1)}} W_{kk'}\; W_{k'k''} \ldots W_{k^{(n-1)}k} \tag{17}$$

where in the above sums $k, k', \ldots, k^{(n-1)}$ vary from k_1 to k_2.

The diagonal elements of W are

$$W_{kk} = \frac{\hbar^2 k^2}{2m} \tag{18, a}$$

and non-diagonal elements

$$W_{kk'} = U_{kk'} = \frac{1}{L} v(k'-k)\; \rho(k'-k) \tag{18, b}$$

The multiple sum for μ_n is divided into subsums $\mu_0^{(n)}, \mu_1^{(n)}, \ldots\ \mu_n^{(n)}$ where $\mu_j^{(n)}$ is the sum of those terms which have j diagonal elements and $n-j$ non-diagonal elements

$$\mu_n = \mu_0^{(n)} + \mu_1^{(n)} + \ldots + \mu_n^{(n)} \tag{19}$$

i.e.

$$<\mu_n> = <\mu_0^{(n)}> + <\mu_1^{(n)}> + \ldots <\mu_n^{(n)}> \tag{20}$$

μ_n consists of terms of the kind .

$$\sum_{k,\, k', \ldots, k^{(n-j-1)}} (W_{kk})^j\, U_{kk'}\, U_{k'k''} \ldots U_{k^{(n-j-1)}k}$$

$$= \sum_{k,\, k', \ldots, k^{(n-j-1)}} \left(\frac{\hbar^2 k^2}{2m}\right)^j \frac{1}{L^{n-j}}\; v(k'-k) \ldots v(k-k^{(n-j-1)}).\;\; \rho(k'-k) \ldots \rho(k-k^{(n-j-1)}).$$

The average of these terms are

$$\sum_{k,\,k',\dots k^{(n-j-1)}} \left(\frac{\hbar^2k^2}{2m}\right)^j \frac{1}{L^{n-j}} \cdot v(k'-k)\dots\, v(k-k^{(n-j-1)}) \;.\; <\rho(k'-k)\,\dots\,\rho(k-k^{(n-j-1)})>$$

In the limit $L\rightarrow\infty$

$$\sum_k = \frac{L}{2\pi}\cdot\int dk,$$

then the above expression becomes

$$\frac{1}{(2\pi)^{n-j}}\int\cdots\int\left(\frac{\hbar^2k^3}{2m}\right)^j v(k'-k)\,\dots\, v(k-k^{(n-j-1)})\cdot$$
$$\cdot<\rho(k'-k)\,\dots\,\rho(k-k^{(n-j-1)})>dk\,dk'\,\dots\,dk^{(n-j-1)}$$

Equation (10) gives the $<\rho(k'-k)\dots\rho(k-k^{(n-\nu-1)}>$ as a function of $k, k', \dots, k^{(n-\nu-1)}$ and thus the subsums $<\mu_\nu{}^{(n)}>$ and therefore $<\mu_n>$ can be calculated.

The method is general, and in principle, it can be applied to calculate moment of any order.

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ON A GENERAL TYPE OF EVERYWHERE DIVERGENT TRIGONOMETRICAL SERIES

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1. In 1926, Kolmogoroff (1926) constructed a well-known integrable function whose Fourier series diverges everywhere. His example is, however fairly complicated. Later in 1929, Steinhaus (1929) proved that there is a trigonometrical series with coefficients tending to zero which ~~diverges~~ everywhere. He showed that the series in the simple form :

$$\sum_{k=3}^{\infty} \frac{\cos k(x-\log\log k)}{\log k}$$

diverges for every x. Other examples with the same properties have also been constructed by Hardy-Littlewood (1914, 1931) and Wilton (1934). In this note, we generalize Steinhaus's series into the following

Theorem. *Let $\phi(n)$ and $\lambda(n)$ be two positive and monotonic increasing functions of n, both of which tend to infinity with n If*

(i) $$-n\Delta\lambda(n) < k < \sqrt{2} \quad (n > n_0)$$

for some k, where $\Delta\lambda(n) = \lambda(n)-\lambda(n+1)$ and

(ii) $$\phi(n) = 0(n) \quad (n \to \infty),$$

then the series

$$\sum_{n=1}^{\infty} (\phi(n))^{-1} \cos n\{x-\lambda(n)\}$$

diverges everywhere

If we take $\phi(n) = \log n$ and $\lambda(n) = \log\log n$, then, since $\log n/n \to 0$ and $-n\Delta\lambda(n) = n\{\log\log(n+1)-\log\log n\} \to 0$ and $n \to \infty$, $\phi(n)$ and $\lambda(n)$ satisfy respectively (ii) and (i). Thus, our theorem includes Steinhaus's series as a special case.

2. We are in a position to prove the theorem. Let us denote

$$T_n(x) = \sum_{\nu=n+1}^{n+m_n} (\phi(\nu))^{-1} \cos \nu\{x-\lambda(\nu)\},$$

$$T_n = \sum_{\nu=n+1}^{n+m_n} (\phi(\nu))^{-1},$$

where $\{m_n\}$ is a sequence of positive integers which will be given immediately after.

We have

$$T_n \geqslant m_n\{\phi(n+m_n)\}^{-1}.$$

By taking $m_n = [\phi(n)]$, we see that

$$T_n \geqslant \frac{[\phi(n)]}{\phi\{n+[\phi(n)]\}} \to 1$$

as $n \to \infty$, since $\phi(n) = 0(n)$. At the same time,

$$0 \leqslant T_n - T_n(x) = \sum_{\nu=n+1}^{n+m_n} (\phi(\nu))^{-1}\{1-\cos\nu(x-\lambda(\nu)\}$$

$$= 2\sum_{\nu=n+1}^{n+m_n} (\phi(\nu))^{-1}\sin^2\frac{\nu(x-\lambda(\nu))}{2}$$

$$\leqslant \tfrac{1}{2}\sum_{\nu=n+1}^{n+m_n} (\phi(\nu))^{-1}\{\nu(x-\lambda(\nu))\}^2.$$

Now, if $n < \nu \leqslant n+m_n$, then $\lambda(n) < \lambda(\nu) \leqslant \lambda(n+m_n)$. Hence, if x belongs to the interval

$$I_n\colon \{\lambda(n),\ \lambda(n+1)\},$$

then

$$|x-\lambda(\nu)| \leqslant \lambda(n+1)-\lambda(n) = -\Delta\lambda(n).$$

Thus,

$$0 \leqslant T_n - T_n(x) \leqslant \frac{m_n}{2}\ \frac{1}{\phi(n+1)}\ \{(n+m_n)\Delta\lambda(n)\}^2$$

$$\leqslant \frac{1}{2}\{(n+m_n)\Delta\lambda(n)\}^2$$

$$= \frac{1}{2}\{(n+[\phi(n)])\Delta\lambda(n)\}^2$$

$$< \frac{k^2}{2},$$

which is less than unity, for $n > n_1 \geqslant n_0$ by (i), since $\phi(n) = 0(n)$ as $n \to \infty$. Moreover, for a given $0 < \epsilon < 1-k^2/2$, we can choose a positive integer n_2 such that

$$T_n > 1-\epsilon \quad (n > n_2).$$

It follows that

$$T_n(x) = T_n-(T_n-T_n(x)) > 1-\epsilon-k^2/2 > 0.$$

for $x \,\epsilon\, I_n$ and $n > \max(n_1, n_2)$. This inequality is periodic with period 2π. Since every x belongs mod (2π) to an infinite number of the intervals I_n, the series

$$\sum_{\nu=1}^{\infty} (\phi(\nu))^{-1}\cos\nu\,\{x-\lambda(\nu)\}$$

diverges for every value of x. This proves the theorem.

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LIOUVILLE'S FORMULA FOR THE ABSOLUTE CURVATURE OF A CONGRUENCE IN A RIEMANNIAN V_m

By

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Let V_n be a subspace imbedded in a Riemannian V_m with x^i $(i = 1, 2, \ldots, n)$ as the coordinates of any point P and $g_{ij}\, dx^i\, dx^j$ as the fundamental metric. Let $y^\alpha (\alpha = 1, 2, \ldots, m)$ be the coordinates of any point of V_n, and $a_{\alpha\beta}\, dy^\alpha\, dy^\beta$ the fundmental metric of V_m.

Consider m mutually orthogonal congruences in V_m at any point P on V_n such that one curve of the congruence passes through each point of V_m and hence through each point of V_n. Let $\omega_{\sigma|}$ $(\sigma = 0, 1, 2, \ldots, m-1)$ be the unit tangent vectors to the curves of the congruences that pass through any point P on V_n. Let $\omega^\alpha_{\sigma|}$ and $\omega_{\sigma|\alpha}$ $(\alpha = 1, 2, \ldots m)$ denote the contravariant components and covariant components of $\omega_{\sigma|}$ respectively. Since the congruences are mutually orthogonal, we have the relations

$$\omega_{\sigma|} \cdot \omega_{\delta|} = \delta^\sigma_\delta \tag{1}$$

or

$$a_{\alpha\beta}\, \omega^\alpha_{\sigma|}\, \omega^\beta_{\delta|} = \delta^\sigma$$

$$\omega^\alpha_{\sigma|}\ \omega_{\delta|\alpha} = \delta^\sigma_\delta \tag{2}$$

From the well-known relations concerning mutually orthogonal unit vectors (Weatherburn (1957) § 30) we have

$$\sum_\sigma \omega_{\sigma|\alpha} \omega_{\sigma|\beta} = a_{\alpha\beta} \tag{3}$$

$$\sum_\sigma \overset{\alpha}{\omega}_{\sigma|}\ \overset{\beta}{\omega}_{\sigma|} = a^{\alpha\beta} \tag{4}$$

Let $C : x^i = x^i$ (S) be a curve in V_n passing through the point P. The derived vector of $\omega_{\sigma|}$ along the curve C is given by

$$\frac{\delta \omega^\alpha_{\sigma|}}{\delta S} = \overset{\alpha}{\omega}_{\sigma|;\,i}\ e^i \tag{5}$$

where e^i are the components of the unit tangent vector to C.

Let us denote the projection of $\dfrac{\delta \omega_{\sigma|}}{\delta S}$ on $\omega_{\delta|}$ as $\gamma_{\sigma\delta}$ which is given by

$$\gamma_{\sigma\delta} = \omega_{\sigma|\alpha\,;\,i}\ \overset{\alpha}{\omega}_{\sigma|}\ e^i \tag{6}$$

This can be also expressed as

$$\gamma_{\sigma\delta} = \overset{\alpha}{\omega}_{\sigma|.\,;\,i}\ \omega_{\delta|\alpha}\ e^i$$

Since the derived vector of $\omega_{\sigma|}$ along C is orthogonal to $\omega_{\sigma|}$, we have for all values of σ,

$$\gamma_{\sigma\sigma} = 0.$$

Again we have the identity

$$a_{\alpha\beta}\,\omega^{\alpha}_{\sigma|}\,\omega^{\beta}_{\delta|} = 0, \quad (\sigma \neq \delta) \tag{7}$$

Differentiating intrinsically with respect to C, we have

$$\omega_{\sigma|\alpha;\,i}\,\omega^{\alpha}_{\delta|}\,e^{i} + \omega_{\delta|\alpha;\,i}\,\omega^{\alpha}_{\sigma|}\,e^{i} = 0$$

$$\gamma_{\sigma\delta} + \gamma_{\delta\sigma} = 0 \tag{8}$$

Hence the invariant $\gamma_{\sigma\delta}$ is skew-symmetric in its suffixes.

Also we have

$$\mathrm{E}\cdot\nabla\omega_{\sigma|} = \sum_{\delta}\gamma_{\sigma\delta}\,\omega_{\delta|}$$

or,

$$\omega_{\sigma|\alpha;\,i}\,e^{i} = \sum_{\delta}\gamma_{\sigma\delta}\,\omega_{\delta|\alpha} \tag{9}$$

and

$$\omega^{\alpha}_{\sigma|;\,i}\,e^{i} = \sum_{\delta}\gamma_{\sigma\delta}\,\omega^{\alpha}_{\delta|}\,.$$

The first curvature vector of the congruence $\omega_{\sigma|}$ with respect to C in V_m is given by $\frac{\delta\omega_{\sigma|}}{\delta S}$. Let $K_{\sigma|}$ be the magnitude of the first curvature vector of the congruence with respect to C. This has been called the absolute curvature (Pan, 1952) of the congruence with respect to C. Let $W^{\alpha}_{\sigma|}$ be the unit vector along the derived vector $\frac{\delta\omega_{\sigma|}}{\delta S}$.

From (9), we have

$$\frac{\delta\omega^{\alpha}_{\sigma|}}{\delta S} = K_{\sigma|}\,W^{\alpha}_{\sigma|} = \sum_{\delta}\gamma_{\sigma\delta}\,\omega^{\alpha}_{\delta|}\,.$$

Squaring we have

$$K^{2}_{\sigma|} = \sum_{\delta}(\gamma_{\sigma\delta})^{2}.$$

Elsewhere we have defined an absolute geodesic (Nirmala, 1963) of the congruence as a curve in V_n such that the absolute curvature of the congruence vanishes at each point of the curve.

Hence *the necessary and sufficient conditions that the curve C should be an absolute geodesic with respect to the congruence $\omega_{\sigma|}$ are that*

$$\gamma_{\sigma\delta} = 0, \qquad (\delta = 1, 2, \ldots, m).$$

The number of these conditions reduces to $m-1$ since $\gamma_{\sigma\delta}$ vanishes when $\delta = \sigma$. In virtue of (8) these conditions can also be written as

$$\gamma_{\delta\sigma} = 0 \qquad (\delta = 1, 2, \ldots, m)$$

Let now μ be another congruence of V_m such that one curve of it passes through each point of V_n. Let μ^{α} be the contravariant components of the unit tangent to a curve of the congruence μ at the point P. Hence we can express

$$\mu^{\alpha} = \sum_{\sigma} d_{\sigma|} \overset{\alpha}{\omega}_{\sigma|} \tag{10}$$

where $d_{\sigma|} = \cos \theta_{\sigma|}$, $\theta_{\sigma|}$ being the angle between the vectors μ and $\omega_{\sigma|}$.

Differentiating (10) intrinsically with respect to C, we have

$$\overset{\alpha}{\mu}_{;i} e^{i} = \sum_{\sigma} d_{\sigma|;i}\, e^{i} \overset{\alpha}{\omega}_{\sigma|} + \sum_{\sigma} d_{\sigma|} \overset{\alpha}{\omega}_{\sigma|;i} e^{i}$$

Using (9), this can be written as

$$\begin{aligned} \frac{\delta \mu^{\alpha}}{\delta S} &= \sum_{\sigma} \frac{\delta d_{\sigma|}}{\delta S} \overset{\alpha}{\omega}_{\sigma|} + \sum_{\sigma} d_{\sigma|} \sum_{\delta} \gamma_{\sigma\delta} \overset{\alpha}{\omega}_{\delta|} \\ &= \sum_{\sigma} \frac{\delta d_{\sigma|}}{\delta S} \overset{\alpha}{\omega}_{\sigma|} + \sum_{\delta} d_{\delta|} \sum_{\sigma} \gamma_{\delta\sigma} \overset{\alpha}{\omega}_{\sigma|} \\ &= \sum_{\sigma} \left[\frac{\delta d_{\sigma|}}{\delta S} + (\sum_{\delta} d_{\delta|})\, \gamma_{\delta\sigma} \right] \overset{\alpha}{\omega}_{\sigma|} \end{aligned}$$

The left side is equal to $K_{\mu|} W^{\alpha}_{\mu}$ where $K_{\mu|}$ is the absolute curvature of the congruence μ with respect to C and $W^{\alpha}_{\mu|}$ is the unit vector along the absolute curvature vector of the congruence μ.

Hence we have

$$\overset{2}{K}_{\mu|} = \sum_{\sigma} \left[\frac{\delta d_{\sigma|}}{\delta S} + (\sum_{\delta} d_{\delta|}) \gamma_{\delta\sigma|} \right]^{2} \tag{11}$$

We shall call this as *Liouville's formula for the absolute curvature of a congruence* in a Riemannian V_m on account of its resemblance to Liouville's formula for the geodesic curvature of a curve on a surface in ordinary space.

If the curve C is an absolute geodesic of the congruence μ, we have

$$K_{\mu|} = 0.$$

From (11), it follows that

$$\frac{\delta d_{\sigma|}}{\delta S} + (\sum_{\delta} d_{\delta|}) \gamma_{\delta\sigma} = 0.$$

i.e. $$-\sin \theta_{\sigma|} \frac{\delta \theta_{\sigma|}}{dS} + (\sum_{\delta} \cos \theta_{\delta|}) \gamma_{\delta\sigma} = 0 \tag{12}$$

But we have

$$\sum_{\sigma} \cos^2 \theta_{\sigma|} = 1.$$

Differentiating intrinsically with respect to C,

$$\sum_{\sigma} \cos \theta_{\sigma|} \sin \theta_{\sigma|} \frac{\delta \theta_{\sigma|}}{\delta S} = 0 \tag{13}$$

From (12) and (13) we have

$$\sum_{\sigma,\delta} \cos\theta_{\sigma|} \cos\theta_{\delta|}\gamma_{\delta\sigma} = 0.$$

Hence if curve C is an absolute geodesic of the congruence μ, we have the identity $\sum_{\sigma,\delta} \cos\theta_{\sigma|} \cos\theta_{\delta|}\gamma_{\delta\sigma} = 0$ where $\theta_{1|}, \theta_{2|}, \ldots, \theta_{m|}$ are the angles made by the vector μ with the orthogonal ennuple of V_m at the point P.

This work suggested itself in the course of a perusal of an unpublished thesis by H. S. Nagaratnamma.

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APPLICATION OF MELLIN TRANSFORM FOR THE EVALUATION OF CERTAIN INTEGRALS INVOLVING PRODUCT OF JACOBI POLYNOMIALS

By

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1. In this paper we shall apply Mellin transform for evaluating certain integrals involving product of Jacobi polynomials. We write for integers K and n with $0 < K < n$ and $-1 < \beta < 1$

$$F_n^{(K,\beta)}(x) = \frac{(n)!}{2^{K-\frac{1}{2}} \cdot \Gamma(K-\beta+n+1)}\; x(x^2-1)^{K-\beta} \,.\, P_n^{(K-\beta,\,\beta)}(2x^2-1), \tag{1.1}$$

$$G_n^{(K,\beta)}(x) = \frac{(n-K+1)!}{2^{K-\frac{1}{2}}\Gamma(n+\beta)}(1-x^2)^{K+\beta-1} \cdot P_{n-K}^{(K+\beta-1,\,-\beta)}(2x^2-1), \tag{1.2}$$

$$R_n^{(K,\beta)}(x) = \frac{(n)!}{2^{K-\frac{1}{2}}\Gamma(n+K+\beta)}\; (x^2-1)^{K+\beta-1} \,.\, P_n^{(K+\beta-1,-\beta)}(2x^2-1), \tag{1.3}$$

$$S_n^{(K,\beta)}(x) = \frac{(n-K)!}{2^{K-\frac{1}{2}}\Gamma(n-\beta)}\; x(1-x^2)^{K-\beta} \cdot P_{n-K-1}^{(K-\beta,\,\beta)}(2x^2-1). \tag{1.4}$$

2. We shall first prove that for $u > 1$

$$J(u) = \int_{1/n}^{1} F_n^{(K,\beta)}(u\,v) \cdot G_n^{(K,\beta)}(v)dv = \{2^{2K}(2K!\}^{-1} \cdot (u^2-1)^{2K} \cdot u^{2n-2K-2\beta+1} \tag{2.1}$$

This integral can be written as

$$u^{2\beta} J(u) = \int_{1/u}^{1} (u\,v)^{2\beta} \cdot F_n^{(K,\beta)}(uv) \cdot G_n^{(K,\beta)}(v) \cdot v^{-2\beta}dv. \tag{2.2}$$

This integral can be rewritten as a standard form of a convolution for Mellin transformation [Titchmarch, 1959 or Erdelyi, 1954, 6.1(13)]

$$\int_0^\infty (u\,v)^{2\beta} F_n^{(K,\beta)}(uv) \cdot U(uv-1) \cdot v^{-2\beta} \cdot G_n^{(K,\beta)}(v) \cdot [1-U(v-1)] \cdot dv, \tag{2.3}$$

where $U(x) = 0,\ x < 0;\ = 1,\ x > 0.$ Thus we have

$$\mathfrak{M}\{u^{2\beta} J(u); S\} =$$

$$= \mathfrak{M}\{u^{2\beta} F_n^{(K,\beta)}(u) \cdot U(u-1); S\} \cdot \mathfrak{M}\{u^{-2\beta} G_n^{(K,\beta)}(u) \cdot [1-U(u-1)]; 1-S\}. \tag{2.4}$$

From Rodrigue's formula [Erdelyi, 1953], we have

$$u^{2\beta}F_n^{(K,\beta)}(u)=\frac{u}{2^{K+n-\frac{1}{2}}\Gamma(K-\beta+1)}\left(\frac{d}{udu}\right)^n\{(u^2-1)^{n+K-\beta}\cdot u^{2n+2\beta}\},\tag{2.5}$$

and

$$u^{-2\beta}G_n^{(K,\beta)}(u)=\frac{(-)^{n-K}}{2^{n-\frac{1}{2}}\Gamma(n+\beta)}\left(\frac{d}{udu}\right)^{n-K}\{(1-u^2)^{n+\beta-1}\cdot u^{2n-2\beta-3K}\}.\tag{2.6}$$

Furthermore if the Mellin transformation of a certain function $h(x)$ is $k(S)$ then the Mellin transformation of $\left(\frac{d}{xdx}\right)^n h(x)$ will be $(-)^n.2^n.\left(\frac{S}{2}-n\right)_n\cdot k(S-2n)$. Using these results and the tables Erdelyi, 1954 6.2(31), (32), 6.1(7)] and after some simplification, we have

$$\mathfrak{M}\left\{u^{2\beta}F_n^{(K,\beta)}(u)\cdot U(u-1);\,S\right\}=\frac{\Gamma\left(\frac{1}{2}+n-\frac{S}{2}\right)\cdot\Gamma\left(-n-K-\frac{S}{2}-\frac{1}{2}\right)}{2^{K+\frac{1}{2}}\Gamma\left(\frac{1-S}{2}\right)\cdot\Gamma(1-\beta-S/2)},\tag{2.7}$$

$$\mathfrak{M}\left\{u^{-2\beta}G_n^{(K,\beta)}(u)\cdot[1-U(u-1)];\,1-S\right\}=\frac{\Gamma\left(\frac{1-S}{2}\right)\cdot\Gamma(1-\beta-S/2)}{2^{K+\frac{1}{2}}\Gamma\left(\frac{1}{2}+n-\frac{S}{2}\right)\Gamma\left(\frac{1}{2}-\frac{S}{2}-n+K\right)},\tag{2.8}$$

hence

$$\mathfrak{M}\{u^{2\beta}J(u);S\}=\frac{B\left(-n-K-\frac{S}{2}-\frac{1}{2},\ 2K+1\right)}{2^{2K+1}\cdot(2K)!}.\tag{2.9}$$

But from the tables [Erdelyi, 1954, 6.2(32), 6.1(7)]

$$[2^{2K}(2K)!]^{-1}(u^2-1)^{2K}\cdot u^{2n-2K+1}\cdot U(u-1)$$

has the same transform, hence the formula follows.

3. Next we prove that for $u>1$

$$J_1(u)=\int_{1/u}^{1}R_n^{(K,\beta)}(uv)\cdot S_n^{(K,\beta)}(v)dv=[2^{2K}(2K)!]^{-1}(u^2-1)^{2K}\cdot u^{2n-2K+2\beta-2}.\tag{3.1}$$

As before, this integral can be written in a standard form of a convolution for Mellin transformation

$$u^{-2\beta}J_1(u)=\int_{1/u}^{1}(uv)^{-2\beta}R_n^{(K,\beta)}(uv)\cdot v^{2\beta}\cdot S_n^{(K,\beta)}(v)dv$$

$$=\int_0^\infty(uv)^{-2\beta}R_n^{(K,\beta)}(uv)\cdot U(uv-1)\cdot v^{2\beta}S_n^{(K,\beta)}(v)\cdot[1-U(v-1)]\,dv,\tag{3.2}$$

where $U(x)=0$, for $x<0$; $=1$, for $x>0$.

Thus we have

$$\mathfrak{M}\{u^{-2\beta}J_1(u);\, S\} = \mathfrak{M}\left\{u^{-2\beta}R_n^{(K,\,\beta)}(uv)\cdot U(u-1);\, S\right\}\cdot\mathfrak{M}\left\{u^{2\beta}S_n^{(K,\,\beta)}(u)[1-U(u-1);\,1-S\right\}.$$

From Rodrigue's formula [Erdelyi, 1953], we have (3.3)

$$u^{-2\beta}R_n^{(K,\,\beta)}(u) = \frac{1}{2^{n+K-\frac{1}{2}}\Gamma(K+n+\beta)}\left(\frac{d}{udu}\right)^n\{(u^2-1)^{n+K+\beta-1}\cdot u^{2n-2\beta}\}, \tag{3.4}$$

$$u^{2B}S_n^{(K,\,\beta)}(u) = \frac{(-)^{n-K-1}\cdot u}{2^{n-3/2}\,\Gamma(n-\beta)}\left(\frac{d}{udu}\right)^{n-K-1}\{(1-u^2)^{n-\beta-1}\cdot u^{2n-K2+2\beta-2}\} \tag{3.5}$$

Using these results and the tables [Erdelyi, 1954, 6.2(31), (32), 6.1(7)] and after some simplification, we have

$$\mathfrak{M}\left\{u^{-2\beta}R_n^{(K,\,\beta)}(u)\cdot U(u-1);S\right\} = \frac{\Gamma\left(n+1-\frac{S}{2}\right)\cdot\Gamma\left(1-n-K-\frac{S}{2}\right)}{2^{K+\frac{1}{2}}\Gamma\left(1-\frac{S}{2}\right)\Gamma\left(1-\frac{S}{2}+\beta\right)} \tag{3.6}$$

$$\mathfrak{M}\left\{u^{2\beta}S_n^{(K,\,\beta)}(u)\cdot[1-U(u-1);\,1-S\right\} = \frac{\Gamma(1-S/2)\cdot\Gamma\left(1+\beta-\frac{S}{2}\right)}{2^{K+\frac{1}{2}}\Gamma\left(n+1-\frac{S}{2}\right)\cdot\Gamma\left(2-n+K-\frac{S}{2}\right)}. \tag{3.7}$$

Hence

$$\mathfrak{M}\{u^{-2\beta}J_1(u),\, S\} = \frac{B(1-n-K-S/2,\, K+1)}{2^{2K+1}(2K)!} \tag{3.8}$$

But from the table [Erdelyi, 1954, 6.2(32), 6.1(7)]

$$[2^{2K}(2K)!]^{-1}(u^2-1)^{2K}\cdot u^{2n-2K-2}U(u-1)$$

has the same Mellin transform; hence the formula follows.

4. If in (2.1) and (3.1) we make the substitutions

$$u^2v^2-1 = x(u^2-1),\ z = u^2-1,$$

after some simplification we get

$$(1+z)^n = \frac{(2K)!(n)!(n-K+1)!}{\Gamma(K-\beta+n+1)\cdot\Gamma(n+\beta)}$$

$$\times\int_0^1 x^{K-\beta}(1-x)^{K+\beta-1}P_n^{(K-\beta,\,\beta)}(2xz+1)\cdot P_{n-K-1}^{(K+\beta-1,-\beta)}\left\{\frac{(2x-1)z+1}{z+1}\right\}dx, \tag{4.1}$$

$$(1+z)^n = \frac{(2K)!\cdot(n)!\cdot(n-K)!}{\Gamma(n+K+\beta)\,\Gamma(n-\beta)}$$

$$\times\int_0^1 x^{K+\beta-1}\cdot(1-x)^{K-\beta}\cdot P_n^{(K+\beta-1,\,-\beta)}(2xz+1)\cdot P_{-K-1}^{(K-\beta,\,\beta)}\left\{\frac{(2x-1)z+1}{z+1}\right\}dx. \tag{4.2}$$

These were the integrals to be evaluated.

5. Particular cases : If in (4.1) and (4.2) we write $\beta = \frac{1}{2}$, we get

$$(z+1)^n = \frac{2^{4K-2}\{\Gamma(K)\}^2(2K)!(2n+1)!(2n-2K)!}{\pi\, \Gamma(2n) \cdot \Gamma(2n+2K+1)}$$

$$\times \int_0^1 \frac{\{x(1-x)\}^{K-\frac{1}{2}}}{(1+xz)^{\frac{1}{2}}} C^K_{2n+1}\{(xz+1)^{\frac{1}{2}}\}\, C^K_{2n-2K}\left\{\left(\frac{xz+1}{z+1}\right)^{\frac{1}{2}}\right\} dx, \tag{5.1}$$

$$(1+z)^{n-\frac{1}{2}} = \frac{2^{4K-2}\{\Gamma(K)\}^2(2K)! \cdot (2n)! \cdot \Gamma(2n-2K)}{\pi \cdot \Gamma(2n-1) \cdot \Gamma(2n+2K)}$$

$$\times \int_0^1 \frac{\{(1-x)\}^{K-\frac{1}{2}}}{(1+xz)^{\frac{1}{2}}} C^K_{2n}\{(xz+1)^{\frac{1}{2}}\} \cdot C^K_{2n-2K-1}\left\{\left(\frac{xz+1}{z+1}\right)^{\frac{1}{2}}\right\} dx. \tag{5.2}$$

If we take the standardization $C^0_n(x) = \lim K \to 0\, K^{-1} C^K_n(x)$, we can easily obtain the values of similar integrals involving product of Chebyshev polynomials by writing $K = 0$

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ON THE SECOND ORDER DIFFERENTIAL EQUATION WHICH HAS ORTHOGONAL POLYNOMIAL SOLUTIONS

BY

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1. *Introduction.* For any sequence of orthogonal polynomials $\{P_n(x)\}$, where $P_n(x)$ is of exactly degree n with leading coefficient unity, three consecutive polynomials satisfy a linear recurrence relation

$$P_n(x)-(x-c_n)P_{n-1}(x)+\lambda_n P_{n-2}(x) = 0, \tag{1.1}$$

for $n \geqslant 2$, c_n and λ_n constants. Moreover, the orthogonal polynomial set $\{P_n\}$ satisfies a differential equation of the form

$$AP_n''(x)+BP_n'(x)+C_nP_n(x) = 0, \tag{1.2}$$

where A and B are polynomials in x, independent of n, of degrees not exceeding 2 and 1 respectively, and C_n is a constant depending on n, if, and only if, $\{P_n\}$ is a set of classical orthogonal polynomials, a set of Bessel or related polynomials, or a trivial type ax^n+bx^m (Szegö, 1939).

In this paper we wish to consider other classes of orthogonal polynomials (nonclassical) which satisfy a second order differential equation of the form

$$A_nP_n''(x)+B_nP_n'(x)+C_nP_n(x) = 0, \tag{1.3}$$

where A_n, B_n, and C_n are polynomials in x and n with fixed degrees (independent of n), and, conversely, to find a necessary condition for the polynomial coefficients A_n, B_n, and C_n in order that equation (1.3) have orthogonal polynomial solutions.

2. *A Derivation of a Second Order Differential Equation.* W. Hahn (1940) and H. L. Krall (1938, 1940) have found several fourth order differential equations with nonclassical orthogonal polynomial solutions. Hence it is quite natural to investigate the existence and actual construction of a second order differential equation which has the same polynomial solutions. Such an equation may not exist (as Hahn has indicated in his work), but, as we shall now show, this is not the case for the sets found by Krall.

Consider the differential equation

$$L_4(y_n) = (x^2-1)^2y_n^{(iv)}(x)+8x(x^2-1)y_n'''(x)+(4\alpha+12)(x^2-1)y_n''(x)+8\alpha xy_n'(x) = \lambda_ny_n(x), \tag{2.1}$$

for which Krall found the orthogonal polynomial solutions $\{Y_n\}$, where

$$\int_{-1}^{+1} Y_n(x)\ Y_k(x)\ d\psi(x) = 0 \quad (n \neq k),$$

and

$$\psi(x) = \begin{cases} 1+\alpha & , x = 1; \\ \alpha x & , -1 < x < 1; \\ -(1+\alpha) & , x = -1. \end{cases}$$

The first step is to show that the orthogonal set $\{Y_n\}$ also satisfies a self-adjoint differential equation of the sixth order. Differentiating (2.1) twice, we obtain a second sixth order differential equation which is satisfied by $\{Y_n\}$. By combining the sixth order equations, we obtain another fourth order equation. Finally, we combine this newly derived differential equation with (2.1) to obtain the desired second order differential equation (since the third and fourth derivatives are eliminated simultaneously).

The most general, self-adjoint, sixth order differential equation is

$$L_6(u) = au^{(vi)}+3a'u^{(v)}+bu^{(iv)}+(2b'-5a''')u'''+cu''+(c'-b'''+3a^{(v)})u' = \mu u, \tag{2.2}$$

where a, b, and c are arbitrary polynomials. The formula of Lagrange corresponding to the operator $L_6(u)$ is

$$vL_6(u)-u\mathrm{L}_6(v) = \frac{d}{dx}\,[P(u, v)],$$

where

$$vL_6(u)-uL_6(v) = aw_{06}+3a'w_{05}+bw_{04}+(2b'-5a''')w_{03}cw_{02}+(c'-b'''+3a^{(v)})w_{01},$$

$$P(u, v) = aw_{05}+2a'w_{04}-(2a''-b)w_{03}-(3a'''-b')w_{02}+(3a^{(iv)}-b''+c)w_{01}$$

$$-aw_{14}-a'w_{13}+(3a''-b)w_{12}+aw_{23},$$

and

$$w_{ij} = v^{(i)}u^{(j)}-v^{(j)}u^{(i)}.$$

Forming the Stieltjes integral of $vL_6(u)-uL_6(v)$ with respect to $\psi(x)$, we have

$$\int_{-1}^{+1}[vL_6(u)-uL_6(v)]d\psi(x) = \alpha P(u, v)\Big]_{-1}^{+1}+(vL_6(u)-uL_6(v))\Big|_{x=1,\,-1} \tag{2.3}$$

For $a = (x^2-1)^3$, $b = 60(x^2-1)$, and $c = 12(\alpha^2+31\alpha+90)(1-x^2)+24\alpha$, the right side of (2.3) is zero, and we have

$$\int_{-1}^{+1} (vL_6(u)-uL_6(v))d\psi(x) = 0. \tag{2.4}$$

Moreover, using these values of a, b, and c, and a suitably chosen value of μ (say, μ_n), the equation (2.2) has a polynomial solution of degree n with leading coefficient unity, so that $L(P_n) = \mu_n P_n$.

Now letting $u = P_n$ and $v = P_k$ $(n \neq k)$, (2.4) becomes

$$(\mu_n-\mu_k)\int_{-1}^{+1} P_n\,P_k\,d\psi(x) = 0;$$

hence, $P_n(x) \equiv Y_n(x)$. We conclude that the orthogonal polynomials $\{Y_n\}$ are also solutions of the differential equation

$$(x^2-1)^3y_n{}^{(vi)}+18x(x^2-1)^2y_n{}^{(v)}+60(x^2-1)y_n{}^{(iv)} \tag{2.5}$$

$$-600x(x^2-1)y_n'''+[12(\alpha^2+31\alpha+90)(1-x^2)+24\alpha]y_n''-24(\alpha^2+31\alpha)xy_n' = \mu_n y_n$$

Proceeding to find a second order differential equation, we form the derivatives of (2.1) and combine these equations with (2.5) to obtain the fourth order equation,

$$L_6(Y_n)-(x^2-1)L_4''(Y_n)-2xL_4'(Y_n) = \mu_n Y_n-\lambda_n(x^2-1)Y_n''-2\lambda_n xY_n'.$$

To this equation, we add $4(x^2-1)^{-1}L_4(Y_n)+(96+4\alpha)L_4(Y_n)$ to obtain the desired second order differential equation,

$$(x^2-1)[(4\alpha^2+4\alpha+\lambda_n)x^2-(4\alpha^2-4\alpha+\lambda_n)]Y_n''(x)+2x[(4\alpha^2+4\alpha+\lambda_n)x^2$$
$$-(4\alpha^2-12\alpha+\lambda_n)]Y_n'(x)-[(\mu_n+4\alpha+96)x^2-(\mu_n+4\alpha+96-4\lambda_n)]Y_n(x) = 0,$$

where from (2.1) and (2.5) respectively

$$\lambda_n = n(n+1)(n^2+n+4\alpha-2)$$

and

$$\mu_n = n(n+1)(n^4+2n^3-97n^2-98n+192-372\alpha-12\alpha^2).$$

The resulting equation is of the general form

$$A_nP_n''(x)+B_nP_n'(x)+C_nP_n(x) = 0,$$

where A_n, B_n, and C_n all depend on x and n, and is satisfied by the nonclassical orthogonal set $\{Y_n(x)\}$. Moreover, the method of construction can be applied to other classes of polynomials, given by Krall in (1940).

3. A Necessary Condition

On the other hand, we might find possible restrictions on the differential equation (1.3) in order that it may have orthogonal polynomial solutions. In particular, we wish to find a condition which restricts the polynomial coefficients A_n, B_n, and C_n.

In quest of such a general result, let us consider the recurrence relation (1.1) along with the differential equation (1.3). In what follows, Q with any subscript represents a function of A_n, B_n, C_n, and their derivatives

The differential equation (1.3) is also satisfied by P_{n-1} and P_{n-2} so that we have

$$A_iP_i''(x)+B_iP_i'(x)+C_iP_i(x) = 0 \quad (i = n, n-1, n-2). \tag{3.1}$$

Repeated differentiation of (3.1) yields the following sets of equations :

$$A_iP_i'''+Q_{1,i}P_i''+Q_{2,i}P_i'+C_i'P_i = 0 \quad (i = n, n-1, n-2); \tag{3.2}$$

$$A_iP_i^{(iv)}+Q_{3,i}P_i'''+Q_{4,i}P_i''+Q_{5,i}P_i'+C_i''P_i = 0 \quad (i = n, n-1, n-2); \tag{3.3}$$

$$A_iP_i^{(v)}+Q_{6,i}P_i^{(iv)}+Q_{7,i}P_i'''+Q_{8,i}P_i''+Q_{9,i}P_i'+C_i'''P_i = 0 \quad (i = n, n-1, n-2). \tag{3.4}$$

Also, by successive differentiation of (1.1), we obtain

$$P_n^{(r)}-(x-c_n)P_{n-1}^{(r)}+\lambda_nP_{n-2}^{(r)}-rP_{n-1}^{(r-1)} = 0 \quad (r = 0, \dots, 5). \tag{3.5}$$

Consequently, by treating $P_i^{(r)}$ $(i = n, n-1, n-2,\ r = 0, \dots, 5)$ as unknowns, we see that the equations (3.1) through (3.5) represent eighteen linear homogeneous equations in eighteen unknowns. We conclude that if these equations have a nontrivial solution, then the matrix of coefficients of $P_i^{(r)}$ must be singular; that is, the determinant of the matrix of coefficients must be zero. This establishes the

Theorem. *If a second order differential equation* $A_nP_n''+B_nP_n'+C_nP_n = 0$, *as previously defined, has a set of orthogonal polynomial solutions, then the polynomial coefficients* A_n, B_n *and* C_n *must satisfy an eighteenth order determinant defined by the matrix of coefficients of the above equations.*

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ON SUBSERIES OF CONDITIONALLY CONVERGENT SERIES

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Let Σu_n be a conditionally convergent series of real terms and ξ be any point in (0, 1] expressed uniquely in a non-terminating dyadic scale as

$$\xi = 0 \cdot \xi_1 \xi_2 \ldots \xi_n \ldots \text{ where } \xi_n = 0 \text{ or } 1.$$

A one-one correspondence between the points in (0, 1] and all possible infinite subseries of Σu_n is established by associating to the point ξ the infinite subseries $\sum\limits_{k=1}^{\infty} \xi_k u_k$.

The purpose of the present note is to prove the following theorem. Throughout the note subseries will always stand for infinite subseries of Σu_n.

Theorem : *Each of the following sets is everywhere dense in* (0, 1]

(a) *the set of points ξ for which the corresponding subseries converge,*

(b) *the set of points ξ for which the corresponding subseries diverge to $+\infty$(or $-\infty$),*

(c) *the set of points ξ for which*

$$\operatorname*{Lt}_{n} \sup \sum_{k=1}^{n} \xi_k u_k < +\infty, \quad \operatorname*{Lt}_{n} \inf \sum_{k=1}^{n} \xi_k u_k = -\infty,$$

(*d*) *the set of points ξ for which*

$$\operatorname*{Lt}_{n} \sup \sum_{k=1}^{n} \xi_k u_k = +\infty, \quad \operatorname*{Lt}_{n} \inf \sum_{k=1}^{n} \xi_k u_k > -\infty,$$

(e) *the set of points ξ for which*

$$\operatorname*{Lt}_{n} \sup \sum_{k=1}^{n} \xi_k u_k < +\infty, \quad \operatorname*{Lt}_{n} \inf \sum_{k=1}^{n} \xi_k u_k > -\infty,$$

(f) *the set C of all points ξ for which the corresponding subseries converge to the given number c.*

To prove the theorem we shall require the following lemma.

Lemma : *If Σu_n is a conditionally convergent series then whatever be λ, μ* $(-\infty \leqslant \mu \leqslant \lambda \leqslant +\infty)$, *there is an infinite subseries $\Sigma\, u_{s_i}$ such that*

$$\operatorname*{Lt}_{n} \sup \sum_{i=1}^{n} u_{s_i} = \lambda \quad \text{and} \quad \operatorname*{Lt}_{n} \inf \sum_{i=1}^{n} u_{s_i} = \mu.$$

The lemma can be easily proved by following arguments almost similar to those used in proving Riemann's rearrangement theorem (Knopp, 1947).

Proof of the theorem : Let ξ be any point in $(0, 1]$, then ξ can be represented uniquely as

$$\xi = 0{\cdot}\xi_1\xi_2 \dots \xi_n \dots \text{ where } \xi_n = 0 \text{ or } 1.$$

Let $\epsilon > 0$ be arbitrary and select the positive integer m so large that $\frac{1}{2^m} < \epsilon$. Let $\xi_1u_1+\xi_2u_2+\dots+\xi_mu_m+\xi'_{m+1}u_{m+1} = d$, where $\xi'_{m+1} = 1$ or 0 according as $\xi_{m+1} = 0$ or 1. Since the series $\sum_{r=m+2}^{\infty} u_r$ is conditionally convergent,by the lemma, given any $\lambda, \mu(-\infty \leqslant \mu < \lambda \leqslant +\infty)$ there is a subseries $\sum_{i=1}^{\infty} u_{s_i}$ of $\sum_{r=m+2}^{\infty} u_r$ such that

$$\operatorname*{Lt}_{n} \sup \sum_{i=1}^{n} u_{s_i} = \lambda \quad \text{and} \quad \operatorname*{Lt}_{n} \inf \sum_{i=1}^{n} u_{s_i} = \mu.$$

Let $\xi' = 0.\xi_1\xi_2 \dots \xi_m\, \xi'_{m+1}\eta_{m+2}\, \eta_{m+3} \dots \eta_k \dots$ where $\eta_k = 1$ or 0 $(k = m+2, m+3,\dots)$ according as u_k occurs in $\sum_{i=1}^{\infty} u_{s_i}$ or not. Obviously $|\,\xi-\xi'\,| \leqslant 1/2^m < \epsilon$.

By taking $\lambda = \mu$ any finite number ; $\lambda = \mu = +\infty($ or $-\infty)$; $\lambda < +\infty, \mu = -\infty$; $\lambda = +\infty$, $\mu > -\infty$; $\lambda < +\infty$, $\mu > -\infty$ and $\lambda = \mu = c-d$ the proofs of (a), (b), (c), (d), (e) and (f) follow because $\epsilon > 0$ is arbitrary and ξ is any point in $(0, 1]$.

Hill (1942)has shown that the set of points ξ in $(0, 1]$ for which

$$\operatorname*{Lt}_{n} \ \sup \ \sum_{k=1}^{n} \xi_k u_k = +\infty \quad \text{and} \quad \operatorname*{Lt}_{n} \inf \sum_{k=1}^{n} \xi_k u_k = -\infty$$

is a residual set and so it follows that each of the sets with the properties under (a), (b), (c), (d), (e) and (f) is a set of the first category in $(0. 1]$.

The author is thankful to Dr. B. K. Lahiri for his kind help during the preparation of the note.

R. S. N. COLLEGE, BARRACKPORE,
WEST BENGAL, INDIA.

References

Hill, J. D. (1942), Some theorems on subseries, *Bull. Amer. Math. Soc.*, **48**, p 103.
Knopp, K.. (1947), *Theory and application of infinite series*, p. 318.

CALCUTTA MATHEMATICAL SOCIETY

Report of the Council on the state of affairs of the Society for the year 1963.

The Council of the Calcutta Mathematical Society has the pleasure in placing the following reports on the state of affairs of the Society for the year 1963, as required by Rule 25 of the Society's Constitution.

At the outset the Council regrets to state that on account of circumstances beyond control the Annual General Meeting of the Society could not be held within the scheduled date viz., the 31st January, 1964 as required by Rule 30 of the Constitution. This was, however ratified by the members of the Society at a Special General meeting.

The Council. The Council of the Society for the year 1963 consisting of officers and other members elected at the last Annual General Meeting and the Editorial Secretary and a new Secretary owing to resignation of the old Secretary was as follows:

President

R. N. Sen.

Vice-Presidents

A. C. Banerjee
S. K. Basu
S. K. Chakrabarty
B. S. Ray
C. N. Srinivasiengar.

Treasurer

S. C. Ghosh

Secretary

U. R. Burman
(upto November 11, 1963)

A. Gupta
(from November 12, 1963)

Editorial Secretary

M. C. Chaki

Other members of the Council

Ram Ballabh
P. L. Bhatnagar
P. L. Ganguli
Amritabha Gupta
D. N. Mitra
M. K. Paul
N. C. Bose Majumdar
Sakti Kanta Chakrabarty
N. L. Ghosh
A. K. Mitra
D. N. Mukherjee
B. N. Prasad.

General. The normal activities of the Society were carried more or less smoothly throughout the year. A brief report of the same under different heads is as follows:

Publications. In 1963 the Society published two issues of the Bulletin viz., Vo. 53, No. 4 (1961) and Vol. 54, No. 2 (1962). In this connection, the Council thanks the authorities of the Calcutta University for printing some of the Bulletins free of charge and the officers and members of the staff of the Calcutta University Press for their valued services. Also, in order to expedite the publication of the arrears, the Society has entrusted parts of the job to two other presses viz., Eka Press and Sadhana Press. The Society also thanks the officers and staff of these presses for their cooperation.

The Council is happy to announce that the publication of the Second part of the Golden Jubilee Commemoration volume has already been completed.

The following members were on the Board of Editors of the Bulletin of the Society for the year 1963.

BOARD OF EDITORS

N. M. Basu
A. C. Choudhury
B. N. Prasad
R. N. Sen
Ram Behari
S. Gupta
M Ray
B. R. Seth
M. C. Chaki (*Editorial Secretary*)

Exchange of publication. The Bulletins of the Society have been sent, as usual, to the various countries of the world. The exchanges of journals done last year have also been maintained this year. Arrangements are being made to exchange the reputed journal 'Tensor' (Japan) with our Bulletin.

Library. The subscription of journals which are not available on exchange basis has been continued during the year.

Membership. Twenty-one new members (including one life member) have been enrolled during the year under review.

Meetings. During the year under review there have been six meetings of the Council and five ordinary general meetings of the Society which were mainly devoted to the reading of papers to be published in the forthcoming issues of the Bulletin.

N. R. Sen Memorial Fund. It was resolved that "N. R. Sen Memorial Sub-Committee" be formed to advise the Council on this matter, and that the Committee be constituted of the following members, with power to co-opt more members as necessity arises.

Prof. P. K. Bose
DR. R. N. Bhattacharya
Sri U. R. Burman (*Secretary of the Committee*)

The Council also resolved to open a separate Trust Fund designated as the "N. R. Sen Memorial Fund", the Secretary and the Treasurer being authorised to operate on the same.

Finance. This year the accounts of the Society have been audited by Chartered Accountants Messrs. A. M. Paul and Co. as required by the West Bengal Societies Registration Act, 1961, and the Council offer its heartfelt thanks to them for their sincere cooperation in preparing the Final Accounts and Balance Sheet. Thanks are also due to the Internal Auditors' Dr. A. C. Choudhury and Dr. B. K. Lahiri for their initial examination of the Accounts.

The Society received the following grants-in-aid of publication during the year under review.

i) Government of India	Rs. 4,500.00
ii) Government of West Bengal	Rs. 2,000.00
iii) National Institute of Sciences of India	Rs. 1,000.00

The Council is happy to declare that these grants have been utilised for the purposes for which they were made. The Council takes this opportunity to offer its grateful thanks to the Government of India, Government of West Bengal and National Institute of Sciences of India.

Obituary. In October 1963 the Society lost one of its honorary members, the world-renowned Prof. H. Hadamard who died at the age of 98. The Society deeply mourned his death and paid respect to his memory. The Society also condoled the death of another member Sri Sisir Kumar Banerjee.

THE CALCUTTA MATHEMATICAL SOCIETY

92, Acharya Prafulla Chandra Road, Calcutta

AUDIT REPORT : 1963

We have audited the attached Balance Sheet of the Calcutta Mathematical Society as at 31st December 1963, and Income and Expenditure Account for the year ended on that date annexed thereto with the books of accounts, vouchers and other records of the Society produced to us and have obtained all the information and explanations which to the best of our knowledge and belief were necessary for the purpose of our audit.

The said Balance Sheet of the Society is its first Balance Sheet and the same has been drawn up on the basis of the valuation of the assets as on 1-1-63, as valued and certified by the management of the Society which were not capable of independent verification by us. The opening balance of the General Fund represents the total net values of the assets as on 1-1-63. We understand that there is staff provident fund account but the same has not been shown in the Balance Sheet drawn up. This fund account should be brought into the books of the Society at the earliest opportunity.

In our opinion and to the best of our information and explanations given to us the said Balance Sheet of the Society subject to our remarks in the second paragraph of the report and our note in the Income and Expenditure Account, exhibits a true and fair view of the state of affairs of the Society as at the end of its financial year, and the Income and Expenditure Account gives a true and fair view of the Income for its financial year.

Dated, Calcutta,
the 16th March, 1964.

A. M. Paul & Co.
Chartered Accountants

Balance Sheet of The Calcutta Mathematical Society

as at 31st December, 1963

92, Acharya Prafulla Chandra Road, Calcutta

LIABILITIES:	Rs. nP.	Rs. nP.
General Fund:	1,02,741.04	
Add. Surplus being excess of income over expenditure	8,521.52	1,11,262.56
K.K.G.P. Fund:	1,168.08	
Add. Interest during the year	59.50	3,277.58
Cullis Fund:	8,231.06	
Add Interest during the year	239.00	8,470.06
N. R. Sen Memorial Fund:		1,708.00
Bank Charges Payable:		1.25
	Rs.	1,24,669.45

ASSETS:	Rs. nP.	Rs. nP.
Furniture & Fixtures: (as valued by Management)	11,205 00	
Less, Depreciation	1,120.50	10,084.50
Library Books & journals: (As valued by Management)	44,000.00	
Additions during the year	3,634.21	47,634.21
Bulletin Stocks: (As valued by Management)	25,000.00	
Add: Current Stocks	1,500.00	26,500.00
Paper Stocks: (As valued by Management)	3,630 00	
Less, Consumed during the year	570.00	3,060.00
Sundry Debtors: Membership fees outstanding		3,816.00
Investments · (General Fund) (G.P. Notes of face value Rs. 6,000/-)		5,663 72
K.K.G.P. Fund Investments:		
State Bank of India Savings Bank Account	1,290.10	
G.P. Notes (face value Rs. 2,000/-)	1,937.48	3,227.58
Cullis Fund Investments:		
State Bank of India Savings Bank Account	1,912.00	
G. P. Notes (Face Vlue Rs. 8,000/-)	6,558.06	8,470.06
Loan Account: (Golden Jubilee Fund)		431.28
Cash & Bank Balances:		
Cash & Stamps	8.47	
State Bank of India (S.B. A/C.)	6,941.50	
United Bank of India (Current A/C.)	8,411.05	
United Bank of India (Suspense A/C.)	101.00	
P. O. S. B. Account	320.08	15,782 10
	Rs.	1,24,669.45

This is the Balance Sheet referred to in our Audit Report of even date separately attached.

Dated Calcutta.

A. M. PAUL & CO.

THE CALCUTTA MATHEMATICAL SOCIETY

92, Acharya Prafulla Chandra Road, Calcutta

INCOME AND EXPENDITURE ACCOUNT FOR THE YEAR ENDED 31st DECEMBER 1963

	Rs. nP.	Rs. nP.
EXPENDITURE:		
Establishment ..	262.21	
Meeting Expenses ..	267.88	
Binding of Journals ..	419.25	
Printing & Stationery	175.91	
Postage ..	244.43	
Miscellaneous expenses ..	183.74	
Bank charges .	2.90	1,556 32
BULLETIN EXPENDITURES:		
Paper consumed .	570.00	
Establishment	1,544.86	
Provident fund .	18.75	
Paper, Blocks etc. ..	173.75	
Printing ..	170.75	
Postage ..	987.99	
Conveyance ..	60.56	
Bank charges ..	49.59	3,576.25
DEPRECIATION ON FURNITURE ..		1,120.50
EXCESS OF INCOME OVER EXPENDITURE TRANSFERRED TO GENERAL FUND ..		8,521.52
		Rs. 14,774.59

	Rs. nP.	Rs. nP.
INCOME:		
MEMBERSHIP SUBSCRIPTION RECEIVED	893.77	
Outstanding subscription ..	1,824.00	2,717.77
ADMISSION FEE. .		200.00
GOVERNMENT OF INDIA GRANT: ..		4,500.00
INTEREST FROM BANK .		179.60
BULLETIN INCOME·		
Govt. of West Bengal grant ..	2,000.00	
National Institute of Science of India ..	1,000.00	
Sales proceeds of Bulletins	2,677 32	
Stocks of Bulletins (Current) ..	1,500.00	7,177.32
		Rs. 14,774.59

Note:—No Depreciation has been charged on Library books & Journals.

A. M. PAUL & CO.

Chartered Accountants

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92, Acharya Prafulla Chandra Road, Calcutta-9.*

Papers intended for publication in the Bulletin of the Society, and all Editorial correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office; or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of each paper, in the following manner, *viz.*, name or names of authors; year of publication; name of the journal (abbreviation); number of volume; and lastly, the page number. The following would be a useful illustration :

Wilson, B. M (1922), *Proc. Lond. Math. Soc.* (2), **21**, 235.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 50 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

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SOME NEW RESULTS ON THE DISTANCE SET OF CANTOR SET

By

N. C. Bose Majumder, *Calcutta.*

(Received—December 28, 1959.)

Introduction. Steinhaus (1917) proved that the Cantor (1883) set C, though it is non-dense and is of measure zero, possesses the remarkable property that given any $d(0 \leq d \leq 1)$, there exists a pair of points x, y, $x \epsilon C$, $y \epsilon C$ such that $y - x = d$. Or, in other words, the distance set of Cantor set fills the whole interval $0 \leq d \leq 1$. *A* set whose distance set fills a continuum is called an S-set. Steinhaus' (1920) remarkable result that any set of positive interior measure is an S-set may be recalled here. The special feature of the distance set of the Cantor set is that it not only fills a continuum, it fills the whole interval $0 \leq x \leq 1$, in which the Cantor set lies. Such a set has been called an SD-set in our previous paper (1959).

Various proofs of this property of the Cantor set, *viz.*, that it is an SD-set have been given. The first, which is geometrical in nature, was given by Steinhaus (1917) himself. Randolph (1940) later on proved this property arithmetically. Besides proving that the Cantor set was an SD-set, he pointed out that some distances could be realised for more than one pair of points, *i.e.*, there are some d's, $(0 \leq d \leq 1)$, such that the pairs of points (x, y), $x \epsilon C$, $y \epsilon C$, $y - x = d$ is more than one.

Taking cue from this, Sengupta suggested if something definite could be said about every d in $0 \leq d \leq 1$. The author of the present note succeeded in proving some results (1959) which are embodied in the following theorems:

Theorem 1. *Except possibly for an enumerable set of values of* $d (d \epsilon C)$, *the aggregate of pairs of points* x, y, $x \epsilon C$, $y \epsilon C$, *and such that* $y - x = d$, *has the power of the continuum.*

Theorem 2. *For each of an everywhere dense set of values of* $d (d \epsilon cC$, *complement of Cantorset) the aggregate of pairs* (x, y), $x \epsilon C$, $y \epsilon C$ *such that* $y - x = d$, *has the power of the continuum.*

A conjecture was made in the paper (1959) that if a $d \epsilon cC$ is such that

$$d = \cdot\delta_1 1\ \delta_2 1\ \delta_3 1\ \delta_4 1 \ldots,$$

where d contains an infinity of 1's, while δ_{2n+1} contains 2's only or is void and δ_{2n} contains O's only or is void, then there is only one pair (x, y), of points, $x \epsilon C$, $y \epsilon C$, such that $y - x = d$. In partitular the result was proved for $d = \cdot 1111\ldots\ldots = \frac{1}{2}$.

Here $$x (\epsilon C) = \cdot 020202\ldots = \tfrac{1}{4}$$

and $$y (\epsilon C) = 202020\ldots = \tfrac{3}{4}$$

The proof of this result, was given by a geometrical method In the present paper we prove the conjecture made above in a more definite form (this is given in Section 1) *viz*.

The necessary and sufficient condition that $a\ d$, such that $0 \leq d \leq 1$, be expressible as $d = y - x$, where $x \in C$, $y \in C$, for only one pair $(x.\ y)$ of points, is that, d must be of the form $d = \cdot\delta_1 1\ \delta_2 1\ \delta_3 1\ \delta_4 1 \ldots$, where there are infinity of 1's and $a\ \delta_{2n+1}$ is composed of only 2's or is void and $a\ \delta_{2n}$ is composed of only O's or is void.

In section II, of this note we have proved results more definite than what is given in Theorems 1 and 2, mentioned above (1959) and we confirm all the results of the previous paper (1959) and what was claimed therein as lower bound is now found to be precise power in each case.

Lastly an indication has been made regarding the corresponding theorems for the equation $y + x = d$.

SECTION I

Theorem. *A necessary and sufficient condition that a* $d(0 \leq d \leq 1)$ *may be represented by one and only one pair of points* (x, y), *where* $x \in C$, $y \in C$ *and such that* $y - x = d$, *C being the Cantor middle third set, is that* $d = \cdot\delta_1 1\ \delta_2 1\ \delta_3 1\ \delta_4 1 \ldots \delta_{2k-1} 1\ \delta_{2k} 1 \ldots$, *containing an infinite number of* 1's, (δ *being a block O's, and 2's—following the notation of Hurwitz, which may be void) where no* δ_{2k-1} *should contain a zero nor any* δ_{2k} *should contain a two.*

We have see in a preceding paper (1959), 'On the distance set of the Cantor middle third set', that if d is a number representing a left hand end point of a contiguous interval of the Cantor set (*i.e.*, $d = \cdot\delta_1 1$), then the number of pairs of points (x, y), $x \in C$, $y \in C$ and $d = y - x$, is at least $= 3\cdot 2^{{}^0\delta_1}$, where ${}^0\delta_{2k-1}$ = number of zeros in δ_{2k-1}, and the least value of $3\cdot 2^{{}^0\delta_{2k-1}}$ is 3.

If d is a number representing an external point (*i.e.* $d = \cdot\delta_1$, δ_1 being a block containing infinite number of zeros and twos) or a number representing a right hand end point of contiguous interval (*i.e.*, $d = \cdot\delta_1$, δ_1 containing a finite number of 0's and twos), then the number of pairs of the type mentioned above, is, in each case, infinity, (the power being that of the continuum).

Hence there is no $d \in C$, whose representation by (x, y), $x \in C$, $y \in C$, such that $y - x = d$, is unique (neglecting the trivial case $d = 1$).

It has further been shown (1959), that when $d \in cC$ and is such that:

(1) $d = \cdot\delta_1 1\ \delta_2 1\ \delta_3 1\ \delta_4 1 \ldots \delta_{2n-1} 1\ \delta_{2n} 1\ \delta_{2n+1}$, (containing an even number of 1's). Whether δ_{2n+1} is void, finite or infinite the number of pairs of points of the type mentioned above is in each case infinite (the power in each case being c, that of the continuum).

(2) $d = \cdot\delta_1 1\ \delta_2 1\ \delta_3 1\ \delta_4 1 \ldots \delta_{2n-3} 1\ \delta_{2n-2} 1\ \delta_{2n-1} 1\ \delta_{2n}$ (containing an odd number of 1's).

The number of pairs of points of the type mentioned above, corresponding to the cases δ_{2n}, void, finite and infinite are respectively, at least $3.2^{\sum_{k=1}^{n-1}({}^2\delta_{2k+1} + {}^2\delta_{2k})}$, at least $3.2^{\sum_{k=1}^{n}({}^2\delta_{2k-1} + {}^2\delta_{2k}) - 1}$, and infinite with power c, when ${}^2\delta_{2k}$ = the number of 2 s in δ_{2k}. So the number of pairs is never less than 3.

From what we have said above, it is now obvious that our search for a $d(0\leq d\leq 1)$, whose representation by (x, y), $x \in C$, $y \in C$, and $d = y-x$, may be unique must be confined to $d \in cC$, where $d = \cdot\delta_1 1\,\delta_2 1\,\delta_3 1\,\delta_4 1\ldots\delta_{2k-1} 1\,\delta_{2k} 1\ldots$, containing an infinite number of 1's.

Firstly, we propose to show that the condition stated in the theorem, is necessary, i.e., in order that the representation of d may be unique, it is necessary that no δ_{2k-1} should contain a zero and also that no δ_{2k} should contain a two.

We argue on the line as indicated by Utz. (1951).

Let the line $y = x+d$, be such that it meets only one square A_1, of the four squares that remain after the first cross is deleted from the unit square A corresponding to the first step in the formation of the Cantor sets on the two axes.

After deletion, corresponding to the second step in the formation of the Cantor sets on the two axes, let the line be such, that it cuts only one square A_2 of the squares that remain.

Let us assume that this continues to happen for the line $y = x+d$, at every stage of deletion of crosses, *i.e.*, the line cuts one and only one square at every stage..

It therefore follows that the line $y = x+d$, so chosen, will pass through one and only one point (x, y), $x \in C$, $y \in C$ and the condition of our problem is therefore satisfied.

Now, let us write down the value of a d satisfying the above condition, namely that $y = x+d$, cuts at every stage, one and only one square.

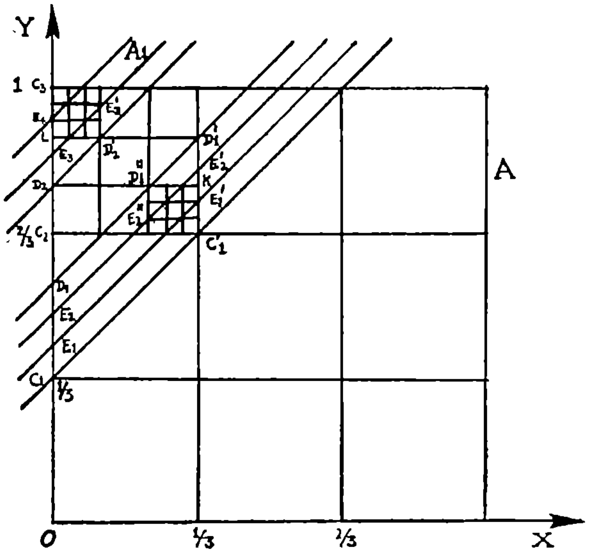


Fig I

In order that $y = x+d(d \in cC)$ may cut only one square A_1 (after the first stage of deletion of a cross) it is necessary that $d > OC_1$, *i.e*, $d > \frac{1}{3}$. Therefore, if $d = \cdot f_1 f_2 f_3\ldots$, where f's are successive digits (0, 1 or 2), then f_1 cannot be zero (∵ all numbers begining with 0 must lie in O and C_1 and a line $y = x + d$ passing through such a point will cut 2 sqq.). Therefore f_1 is either 1 or 2.

(*i*) If $f_1 = 1$ (the line $y = x + d$ cuts the y axis between C_1 and C_2), then, in order that the line may cut only one square A_2 (after the second stage of deletion of a cross) d must lie between C_1 and D_1, and therefore the line $y = x + d$ must cut the right hand vertical line in C'_1 and D'_1—the border points of the sqq. numbering 0 and 1). Therefore f_2 must be 0 or 1 but not 2.

Therefore $\cdot f_1f_2 = \cdot 10$ or $\cdot 11$.

(*ii*) If $f_1 = 2$ (the line $y = x + d$, cuts the y axis between C_2 and C_3), then in order that the line may cut only one square A_2 (after the second stage of deletion of crosses), d must lie between D_2 and C_3, the border points of the sqq. numbering 1 and 2 on the left hand vertical line). Therefore $f_2 = 1$ or 2 but not 0. Therefore $\cdot f_1f_2 = \cdot 21$ or $\cdot 22$.

Therefore we have combining (*i*) and (*ii*), and writing them in their ascending order of magnitudes, $\cdot f_1f_2 = \cdot 10, \cdot 11, \cdot 21, \cdot 22$.

(*a*) Consider now the least one, *viz.*, $\cdot f_1f_2 = \cdot 10$. We are now moving between C'_1 and K and because we want to cut only one sq. A_3 (after the third stage of deletion of crosses) our movements are restricted between C'_1 and E'_1—which is the region of 0 and 1 and not 2 and hence $f_3 = 0$ or 1 but not 2. Therefore $\cdot f_1f_2f_3 = \cdot 100$ or $\cdot 101$.

(*b*) Consider the next higher one, *viz.*, $\cdot f_1f_2 = \cdot 11$.

We are now moving between K and D'_1 and because we want to cut only one square A_3 (after the third stage), our movements are restricted between E'_2 and D'_1, *i e.*, between E'_2 and D'_1, which is the region of 1 and 2 and hence $f_3 = 1$ or 2 but not 0. Therefore $\cdot f_1f_2f_3 = \cdot 111$ or $\cdot 112$.

(*c*) Take the next higher one, *viz.*, $\cdot f_1f_2 = \cdot 21$.

We are now moving between D_2 and L and and because we want cut only 1 sq. A_3 (after the third stage) our movements are restricted between D'_2 and E'_3 which is the region of 0 and 1 but not 2. Hence $f_3 = 0$ or 1 but not 2.

Therefore $\cdot f_1f_2f_3 = \cdot 210$ or $\cdot 211$.

(*d*) Consider finally the last one, *viz.*, $\cdot f_1f_2 = \cdot 22$.

We are now moving between L and C_3 but because we want to cut only 1 sq. A_3 (after the third stage), our movements are restricted between E_4 and C_3 which is the region of 1 and 2 but not 0. Hence $f_3 = 1$ or 2 but not 0.

Therefore $\cdot f_1f_2f_3 = \cdot 221$ or $\cdot 222$.

This prosess of constructing f's is continued indefinitely.

Let us now write down, what we have obtained so far at the successive stages in terms of δ notations:

1st stage:

$$\left.\begin{array}{l} \cdot f_1 = \cdot 1 = \cdot\delta_1 1 \\ \cdot f_1 = \cdot 2 = \cdot\delta_1 \end{array}\right\} \text{Therefore } \delta_1 \text{ does not contain 0, but it may be void.}$$

2nd stage.

$$\left.\begin{array}{l} \cdot f_1 f_2 = \cdot 10 = \cdot\delta_1 1\delta_2 \\ \cdot f_1 f_2 = \cdot 11 = \cdot\delta_1 1\delta_2 1 \\ \cdot f_1 f_2 = \cdot 21 = \cdot\delta_1 1 \\ \cdot f_1 f_2 = \cdot 22 = \cdot\delta_1 \end{array}\right\} \begin{array}{c}\text{Therefore } \delta_1 \text{ does not contain 0 and } \delta_2 \text{ does} \\ \text{not contain 2, but } \delta\text{'s may be void.}\end{array}$$

3rd stage:

$$\left.\begin{array}{l} \cdot f_1 f_2 f_3 = \cdot 100 = \cdot\delta_1 1\delta_2 \\ \cdot f_1 f_2 f_3 = \cdot 101 = \cdot\delta_1 1\delta_2 1 \\ \cdot f_1 f_2 f_3 = \cdot 111 = \cdot\delta_1 1\delta_2 {}'\delta_3 1 \\ \cdot f_1 f_2 f_3 = \cdot 112 = \cdot\delta_1 1\delta_2 1\delta_3 \\ \cdot f_1 f_2 f_3 = \cdot 210 = \cdot\delta_1 1\delta_2 \\ \cdot f_1 f_2 f_3 = \cdot 211 = \cdot\delta_1 1\delta_2 1 \\ \cdot f_1 f_2 f_2 = \cdot 221 = .\delta_1 1 \\ \cdot f_1 f_2 f_3 = \cdot 222 = \cdot\delta_1 \end{array}\right\} \begin{array}{c}\text{Therefore } \delta_1 \text{ and } \delta_3 \text{ do not contain} \\ \text{0 and } \delta_2 \text{ does not contain 2, but} \\ \delta\text{'s may be void.}\end{array}$$

Note: We notice that when we want to prepare any stage from the preceding stage, we start with the lowest number $\cdot f_1 f_2 \ldots$, and go up in ascending order. We first use the region of 0 and 1 (and exclude that of 2) and then we use the region of 1 and 2 (and exclude that of 0), and this goes on alternately.

Hence our rule is: 'Exclude 2' and then 'Exclude 0' for any stage (excepting while preparing the 1st, when our rule is 'Exclude 0'). Let us prepare the 4th stage from the 3rd stage by using this rule.

Taking $\cdot f_1 f_2 f_3 = \cdot\delta_1 1\delta_2$, we have

$$\left.\begin{array}{r} \cdot f_1 f_2 f_3 f_4 = \cdot\delta_1 1\delta_2 0 \\ \text{or } \cdot\delta_1 1\delta_2 1 \end{array}\right\}$$

$$\left.\begin{array}{r} \text{or } \cdot\delta_1 1\delta_2 \\ \cdot\delta_1 1\delta_2 1 \end{array}\right\} \text{'Exclude 2'}$$

Taking next, $\cdot f_1 f_2 f_3 = \cdot\delta_1 1\delta_2 1$, we have,

$$\left.\begin{array}{r} \cdot f_1 f_2 f_3 f_4 = \cdot\delta_1 1\delta_2 1\ 1 \\ \text{or } \cdot\delta_1 1\delta_2 1\ 2 \end{array}\right\}$$

$$\left.\begin{array}{r} \text{or } \cdot\delta_1 1\delta_2 1\delta_3 1 \\ \cdot\delta_1 1\delta_2 1\delta_3 \end{array}\right\} \text{'Exclude 0'.}$$

So on.

This process is continued indefinitely.

Thus, in order that $d = y-x$, $y \epsilon C$, $x \epsilon C$, may have one and only one solution, it is *necessary*, *geometrically*, that the line $y = x+d$, must cut one and only one square after every stage of deletion of crosses in the formation of the Cantor sets on the two axes.

And according to the demonostrations made above, *analytically*, *the condition necessary* for the unique solution of $d = y-x$, $y \epsilon C$, $x \epsilon C$, is that $d \epsilon cC$ and $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots\delta_{2k-1}1\delta_{2k}1\ldots$, containing infinite number of 1's, where no δ_{2k-1} should contain a zero and no δ_{2k} should contain a two, one or more δ's may be void.

Hence the first part of our theorem is proved.

To prove now, that *the condition is sufficient*. We now propose to prove that if, $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots\delta_{2k-1}1\delta_{2k}1\ldots$ ad inf., where no δ_{2k-1} contains a zero and no δ_{2k} contains a two, then there is one and only one pair of points (x, y), $x \epsilon C$, $y \epsilon C$ and $y-x = d$.

In order to prove this, it is enough to show that the above line $y = x+d$ intersects one and only one sq. after every stage of deletion of crosses in the formation of Cantor sets on the two axes.

Proof. Let us first see how the group $G_k \, (= \delta_{2k-1}1\delta_{2k}1)$ guides the line $y = x+d$ in cutting sqq. after successive stages of deletion of crosses, where δ_{2k-1} contains only 2's (or it may be void), and δ_{2k} contains only 0,s (or it may be void).

In what follows, if should always be remembered that '0' means bottom square, '1' means middle square and '2' means the top square along side a vertical border line at the rth stage where 0 (or 1 or 2) is the rth digit f_r in $\cdot f_1 f_2 f_3 \, . \, (= d)$.

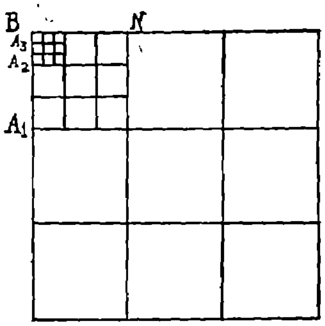


FIG. II.

Since δ_{2k-1} contains only 2's, therefore G_k leads $y = x+d$ to cut the left hand vertical line in $A_1 B$, then in $A_2 B$, then in $A_3 B$ and so on, at successive stages of deletion of crosses and therefore cuts only one square at every such stage. We now arrive at 1; following δ_{2k-1} (if δ_{2k-1} is void, we directly start with this 1).

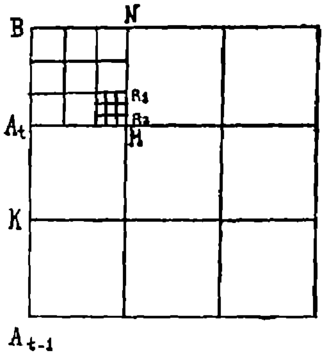


FIG. III.

Therefore, at this stage G_k is leading the line $y = x+d$, so as to cut the left hand vertical line in KA_t, in such a manner that it meets one and only one square $A_t MNB$ and since after this stage, we reach δ_{2k} containing only 0's (or it may be void), therefore the line cuts the right hand vertical line MN in MR_1, then in MR_2 and so on (which are all 0-sqq. *i.e.*, bottom sqq.) at successive stages of deletion of crosses and therefore cuts one and only one sq. at every such stage. We now arrive at 1 following δ_{2k} (if δ_{2k} is void, we come to this 1 after the 1 following δ_{2k-1}).

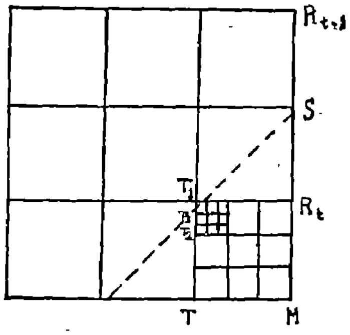


FIG. IV.

At this stage G_k is leading the line $y = x+d$, so as to cut the right hand vertical line in R_tS and cuts only and sq. MR_tT_1T and we arrive at δ_{2k+1} belonging to the group G_{k+1} $(= \delta_{2k+1}1\delta_{2k+2}1)$ containing 2's only and therefore cuts the left hand vertical line T_1T somewhere between T_1T_2 and then somewhere between T_1T_3 and so on. Thus we find that G_k leads $y = x+d$ in such a manner that at every stage it cuts one and only one sq. and this is true for all k $(= 1, 2, 3, \ldots\infty)$.

Therefore $y = x+d$ cuts at every stage one and only one sq. and therefore passes through one and only one point (x, y), $x \epsilon C$, $y \epsilon C$.

That is, if, $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots\delta_{2k-1}1\delta_{2k}1\ldots$ad inf. then there is one and only one point (x, y), $x \varepsilon C$, $y \varepsilon C$ such that $y-x = d$, if δ_{2k-1} does not contain a 0 and δ_{2k} does not contain a 2.

Hence the condition is sufficient.

Note. It is to be noticed that at all stages, whenever we are passing a '1', following a δ_i we are thrown from one vertical line (the extreme left hand border) to the other vertical line (the extreme right hand border of a sq.); *e.g.*, when considering δ_1, we are on the y-axis; as we pass '1' following δ_1 we go to the vertical line (|| to y-axis), the right hand border of the sq. we are considering at this stage; as we now finish considering δ_2 on this line and pass '1' following δ_2, we are back to the left hand boundary of a square. Moreover when we are on the left hand vertical line and are playing with 1 and 2 we cut one and only one square and when we are on the right hand vertical line and are playing with 0 and 1, we cut one and only one square.

Corollary. A necessary condition that a 'd' can have one and only one representation as $d = y-x$, $y \varepsilon C$, $x \varepsilon C$ is that $d>\frac{1}{3}$

For, from what we have discussed above, such a 'd' is given by $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots$ ad inf.

$$= \cdot 100\ldots01100\ldots01100\ldots011\ldots \text{ad inf.}\ldots(1)$$

the block of '0's may contain any number of them as we like. (Since δ_{2k-1} contains no zeros and also no 2, because we want to make d as small as possible; hence δ_{2k-1} is chosen void for all k. Also δ_{2k} contains no 2, but it may contain any number of zeros—in fact more zeros we choose, the smaller becomes the number d.)

Therefore from (1), it is evident that such a 'd' satisfies the inequality, $d>\frac{1}{3}$ always. Hence $d>\frac{1}{3}$ is a necessary condition for its unique representation as $d = y-x$, $y \varepsilon C$, $x \varepsilon C$.

It is moreover evident from (1), that $d = \frac{1}{3}$ is a point of condensation (*i.e.*, a point of unenumerability) from above, of the set of d's, each having the property of uniquely representable as $d = y-x$. $y \varepsilon C$, $x \varepsilon C$.

It is now found that:

(i) each $d(0 \leq d \leq \frac{1}{3})$ must have more than one representation as $y-x = d$, $y \varepsilon C$, $x \varepsilon C$;

and (ii) each $d>\frac{1}{3}$, may have one or more such representations.

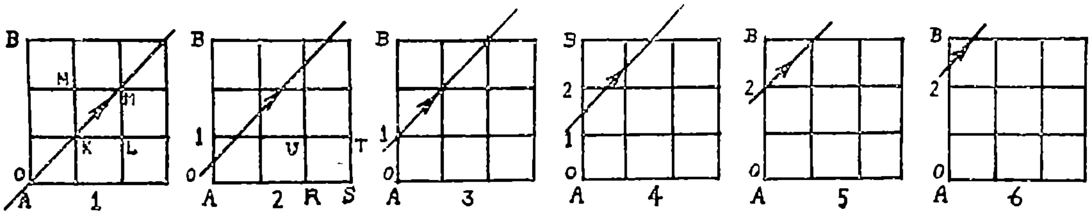
These results are also obvious geometrically.

Section II

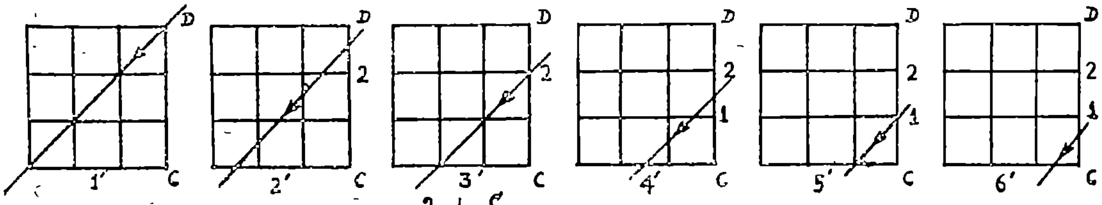
In our earlier paper 'On the distance set of the Cantor middle third set' (1959), we have given some results, in some of which we concluded that our formulae gave the lower bounds of the possible number of pairs (x, y), $y-x = d$, $y \varepsilon C$, $x \varepsilon C$, where $0 \leq d \leq 1$ (*e.g.*, if $d = \delta_1 1$, the number of pairs of points (x, y), $x \varepsilon C$, $y \varepsilon C$, $y-x = d$ is at least 3.2^{δ_1}, thus giving the lower bound of the possible number of such pairs.)

In this section, we propose to deal with the same problem which we now consider in the light of a new method. We shall now show that, not only, all our earlier conclusions are correct, but our formulae at various stages give precisely the number of pairs (x, y), $y-x = d$, $y \in C$, $x \in C$, $0 \leq d \leq 1$, and not merely the lower bounds as was then claimed under the prevailing circumstances.

Before we start considering our proposed problems, we first discuss the behaviour of the line $y = x+d$, where $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1 \ldots$, with respect to the sqq. generated at the various stages of deletion of crosses in the process of constructing the Cantor sets on the x and y-axes. Firstly, we observe that the digits in $\delta_{2k-1}1$ play on left hand vertical lines (like AB figs. V, 1st set) of sqq., whereas the digits on $\delta_{2k}1$ play on right hand vertical lines (like CD figs V 2nd set) of squares (See Note, on page 8).



1st Set



2nd. Set

FIG. V.

In what follows, we shall call a middle sq. like $KLMN$, which cannot give rise to any new square in future stages of deletion of crosses (and therefore contains no Cantor points, excepting the corner points) a β-square, and a corner square like $RSTU$, which will give rise to more sqq. at subsequent stages of deletion of crosses (and therefore contains infinite number of Cantor points), we shall call a γ-square.

It will now be evident from the above 12 figures (figs. V) that if at any stage in δ_{2k-1}, d stops at $a2$ [(figs. V 5)], we shall have arrived at a γ-sq and then we begin to get (imagine, that, we attach an infinite number of zeros after the final 2—corresponding to the infinite number of successive stages of deletion of crosses succeeding the stage of final 2 in δ_{2k-1}), 2 γ-sqq. [(fig. V 1)] at each successive stage, coming out of each γ-sq. of the previous stage, which the line $y = x+d$ may have cut, thus pointing out the existence of at least 2 points corresponding to each such 2 sqq. Therefore these infinite number of zeros give rise to infinite number of pairs of sqq. and therefore infinite number of points with the power c of the continuum.

If d stops at 1 in $\delta_{2k-1}1$ [(fig. V 3)], then at all successive stages only the same 3 points keep on coming-because at no stage, the line can cut any γ-sq. in points besides these 3 points.

If d stops at a 2 in δ_{2k} [(figs. V 3′), then at all successive stages, only the same 3 points keep on coming-because at no stage, the line can cut any γ-sq. in points besides these three.

If d stops at the 1 in $\delta_{2k}1$ [(fig. V 5′)], we shall have arrived at a γ-sq. and then we begin to get (imagine, that, we attach an infinite number of zeros after the final 1, corresponding to the infinite number of successive stages of deletion of crosses succeeding the stage of final 1 in $\delta_{2k}1$),—two γ-sqq. [(figs. V 1′)] at each successive stage, coming out of each γ-sq. of the previous stage, which the line $y = x+d$ may have cut, thus pointing out the existence of at least 2 points corresponding to each such pair of 2 sqq. It should be noticed that when we write any number of zeros after 1 in $\delta_{2k}1$, we consider 1 in $\delta_{2k}1$ to have been passed and these new zeros belong to δ_{2k+1} and therefore we consider now the set of 1st 6 figures of which we use only the fig. 1.

Now, to be definite, let $\delta_{2k-1}1 = 20201$ and let us find out how many new sqq. we generate by this portion only of d. Now first 2 means [(figs. V 6)[one γ-sq. to be associated with 2 new γ-sqq corresponding to the next 0 [(figs. V 2)] to be associated with one new γ-sq., corresponding to the next 2 [(figs. V 6)] to be associated with 2 new γ-sqq. corresponding to the next 0 [(figs. V 2)] to be associated with one new γ-sq. corresponding to 1 [(figs. V 4).] Thus in all we get 4 sqq. from this part of d, thus indicating the existence of 4 corresponding points (x, y), $x \in C$, $y \in C$, $y-x = d$.

Now, we consider $\delta_{2k}1 = 02201$ say. First 0 means 1 γ-sq. [(figs. V6′)] to be associated with 2 new γ-sqq. corresponding to the first 2 [(figs .V 2′)] to be associated with 2 new γ-sqq. corresponding to the 2nd 2 [(figs. V 2′)] to be associated with 1 new γ-sq. corresponding to the second 0 [(figs. V 6′)] to be associated with 1 new γ-sq. corresponding to the final 1 [(figs. V 4′)]. Thus in all, we have collected 4 sqq. from this part of d-indicating the existance of 4 points of the required type.

We are now in a position to consider the results of our previous paper [(1959); mentioned in the introduction as theorems 1 and 2,], in which we discussed them in two parts I and II and then in three parts a, b, c under the head I and in three parts A, B, C under the head II.

$I(a)$. Let $d = \cdot\delta_1$, consisting of finite number of 0's and 2's and ending with a 2 (which represents a right hand end point of a contiguous interval of the Cantor set C).

From what we have seen above, each 0 in it gives two new γ sqq. and each 2 in it gives a new γ-sq., at each of their corresponding stages and finally because the final 2 is supposed to be followed by an infinite number of 0's, we have for each such 0 a pair of new γ-sqq., we observe that there exist an infinite number of pairs of points (x, y), $x \in C$ $y \in C$, $y-x = d$) having the cardinal number c of the continuum which confirms the conclusion made in the previous paper (1959).

$I(b)$. Consider now $d = \cdot\delta_1 1$. (This represents a left hand end point of a contiguous interval).

By what we have said above, we observe, that each 0 in δ_1 generates two new γ-sqq., each 2 in it generates one γ-sq. and the final 1 generates 3 points (corner) and at all subsequent stages, we continue to get the same 3 points. Hence associating all these cases, the number of pairs of points each satisfying the condition of our problem is precisely $3.2^{{}^{0}\delta_1}$. where ${}^{0}\delta_1$ = number of 0's in δ_1; and this confirms our previous conjecture (1959).

I(c). Consider now $d = {}^{\cdot}\delta_1$, where δ_1 contains an infinite number of 0's and 2's that is d is the distance from the origin, of an external point.

Giving the argument as in (*a*), the number of pairs of points satisfying the condition of our problem is infinity and the set of these pairs has cardinal number c, which agrees with our result in the previous paper (1959)

II(A). Let $d = {}^{\cdot}\delta_1 1 \delta_2 1 \delta_3 1 \delta_4 1 \ldots \delta_{2n-1} 1 \delta_{2n} 1 \delta_{2n+1}$, containing an even number of 1's.

(A)i. Let δ_{2n+1} be void, *i.e* let $d = {}^{\cdot}\delta_1 1 \delta_2 1 \ldots \delta_{2n-1} 1 \delta_{2n} 1$.

We have seen before that when we arrive at any group $(\delta_{2k-1} 1 \delta_{2k} 1)$, we play first on lefthand vertical lines while we consider $(\delta_{2k-1} 1)$, that is our figures of reference will be the first set of 6 diagrams (figs. V) and then we go to play on right hand vertical lines while we consider $(\delta_{2k} 1)$ and our figures of reference will be the second set of 6 diagrams (figs. V).

Hence, for our present d, we finally arrive at $\delta_{2n} 1$ and therefore we play on a right hand vertical line till we arrive at the stage 1 [(figs. V 5′,)], which places us on the diagonal of a γ-sq., and because after this we can attach infinite number of 0's (which may be considered to belong to δ_{2n+1}—and we are therefore on a lefthand vertical line) each of which generates a pair of γ-sqq. Hence we get an infinite number of pairs of points satisfying the condition of the problem and the set of these pairs has the power c. This confirms our previous result (1959).

(A)ii. Let δ_{2n+1} be non-empty and finite and therefore ends with a 2.

Therefore $d = {}^{\cdot}\delta_1 1 \delta_2 1 \ldots \delta_{2n-1} 1 \delta_{2n} 1 \;\; \delta_{2n+1}$

In continuation of what we have said in *(A)i*, we have after the stage 1 in $(\delta_{2n} 1)$, pass on to a lefthand vertical line and start playing with δ_{2n+1} until we finally arrive at the last 2, which places us on the diagonal of a γ-sq. [(figs. V 5)]. Now following the same argument as is given in *(A)i*, we prove the existence of infinite number of pairs of points (satisfying the condition of our problem) having the power c, thus agreeing with our previous result (1959).

(A)iii. Let δ_{2n+1} be an infinite complex of 0's and 2's.

As each 0 in it gives two γ-sqq. and each 2 in it, one γ-sq., therefore (as there are an infinite number of zeros in δ_{2n+1}) by associating these sqq. among themselves and also with all the sqq. obtained at the previous stages, we have infinite number of sqq. and therefore infinite number of pairs of points satisfying the condition of the problem, having the power c, which confirms our previous conclusion (1959).

II(B). Let d contain an odd number of 1's. Then d is of the form, $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots\delta_{2n-3}1\delta_{2n-2}1\delta_{2n-1}1\delta_{2n}$.

(B)i. Let δ_{2n} be void, *i.e.*, let $d = \cdot\delta_1 1\delta_2 1\ldots\delta_{2n-3}1\delta_{2n-2}1\ \delta_{2n-1}1$. From what we have said before, it is seen that, when we consider $(\delta_{2k-1}1\delta_{2k}1)$ and are playing with $\delta_{2k-1}1$ (on a left hand vertical line), we get by association $2^{{}^0\delta_{2k-1}}$ γ-sqq. ($\because$ each 0 in δ_{2k-1} generates 2γ-sqq., each 2 in it 1γ-sq., and 1 in it 1 γ-sq.) and then when we are playing with $\delta_{2k}1$ (on a right hand vertical line) we get by association $2^{{}^2\delta_{2k}}$ γ-sqq. ($\because$ each 2 in δ_{2k} generates two γ-sqq, each 0 in it one γ-sq. and 1 in it 1 γ-sq., if it is not the final 1). Therefore, by associating $(\delta_{2k-1}1)$ with $(\delta_{2k}1)$ we get in all $2^{({}^0\delta_{2k-1}+{}^2\delta_{2k})}$ γ-sqq. for all k. But when we arrive at $(\delta_{2n-1}1)$, (we are now on a left vertical line), we get first $2^{{}^0\delta_{2n-1}}$ γ-sqq., but 1 in it, being the final 1, now gives 3 points [corner; figs. V 3], and these same 3 points will continue to appear at all successive stages (as corner points of subsequent γ-sqq.). Therefore associating all these results, we get pairs of points, satisfying the condition of our problem, the number of which is precisely

$$= 3.2^{\sum\limits_{k=1}^{k=n}({}^0\delta_{2k-1}+{}^2\delta_{2k-2})}$$, which confirms the conjecture of our previous paper (1959).

(B)ii. Let δ_{2n} be non-empty and finite, that is

let $$d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots\delta_{2n-3}1\delta_{2n-2}1\ \delta_{2n-1}1\delta_{2n}.$$

With the same argument as given in *(B)i*, we have, when we have finished considering, upto $\delta_{2n-1}1$, the number of γ-sqq. and therefroe the number of pairs of points, so far collected $= 2^{\sum\limits_{k=1}^{n}{}^0\delta_{2k-1}+\sum\limits_{k=1}^{n-1}{}^2\delta_{2k}}$. We now go to consider δ_{2n}, which ends, with *a*2, for which we are playing on a right hand vertical line. Each 0 in it generates one γ-sq. and each 2 in it, excepting the last one, gives a pair of γ-sqq. But when we arrive at the last 2 [(figs. V 3′)], we pass through 3 corner points (of 3 γ-sqq.) and we shall continue to pass through the same three points (corner points of 3γ-sqq.) of γ-sqq., generated at all subsequent stages.

Therefore associating all these results, we have now the pairs of points, satisfying the condition of the problem, the number of which is precisely

$$= \left(2^{\sum\limits_{k=1}^{n}{}^0\delta_{2k-1}+\sum\limits_{k=1}^{n-1}{}^2\delta_{2k}}\right)(2^{{}^2\delta_{2n}-1}.3) = 3.2^{\sum\limits_{k=1}^{n}({}^0\delta_{2k-1}+{}^2\delta_{2k})-1},$$

confirming the cojecture, we made, in our previous paper (1959).

(B)iii. Let δ_{2n} be an infinite complex of 0's and 2's,

i.e., $$d = \cdot\delta_1 1\delta_2 1\ \delta_3 1\delta_4 1\ldots\delta_{2n-3}1\delta_{2n-2}1\ \ \delta_{2n-1}1\delta_{2n}.$$

After we have finished considering $\delta_{2n-1}1$, we now go to play with δ_{2n} on a right hand vertical line. As in the preceding case, each 0 in δ_{2n} generates one γ-sq. and each 2 in it generates a pair of γ-sqq.; associating all these results among themselves, and also the results obtained at the preceding stages, we have ($\because$ there are an infinite number of 2's in δ_{2n}) infinite number of γ-sqq. and therefore correspondingly infinite number of pairs of points, satisfying the condition of our problem, having the power c; and this confirms the result of our previous paper (1959).

II(C). Let d contain an infinity of 1's. Then d is of the form, $d = \cdot\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots \delta_{2k-1}1\delta_{2k}1\ldots$, containing the group ($\delta 1\delta 1$) in unending succession.

(C)i. If an infinite number of δ_{2n-1}'s contain 0's or (and) an infinite number of δ_{2n}'s contain 2's, then, following arguments given in the preceding cases, we have an infinite number of pairs of points, satisfying the condition of our problem, having the power c; this confirms the result of our previous paper (1959).

(C)ii. If at best a finite number of δ_{2n-1}'s have 0's and at best a finite number of δ_{2n}'s have 2's, then arguing as in the preceding cases, we have pairs of points, satisfying the condition of our problem, the number of which is precisely $= 2^{\Sigma('\delta_{2n-1}+'\delta_{2n})}$, thus confirming our result in the previous paper (1959).

(C)iii. Let d be such that no δ_{2n-1} contains a 0 and no δ_{2n} contains a 2.

Consider ($\delta_{2n-1}1\ \delta_{2n}1$). Since δ_{2n-1} contains only 2's (it may be void), therefore, when we play with ($\delta_{2n-1}1$) on a left hand vertical line, we get 1γ-sq. for each 2 [(figs V 6)] and 1γ-sq for 1 [(figs. V 4)]; hence associating we get only 1γ-sq; then we pass on to a right hand vertical line and start playing with ($\delta_{2n}1$). For each 0 in it (δ_{2n} may be void), we get 1γ-sq. [(fig V6′)] and 1γ-sq. for 1 [(figs. V 4′)]; therefore associating we get only 1γ-sq. Hence associating the result for ($\delta_{2n-1}1$) with that for ($\delta_{2n}1$) we get only 1γ-sq. This continues at all stages and therefore we get only 1γ-sq.(whose size is diminishing indefinitely) even after the deletion of crosses infinite number of times and therefore, finally, we get precisely 1 pair of points satisfying the condition of our problem. This confirms our result, obtaind in our previous paper (1959).

Note 1. It is obvious that similar arguments with slight modification may be advanced for the solutions of $y+x=d$, for all possible d's discussed in sections I and II. For instance, we may enunciate, that, 'A necessary and sufficient condition that a d may be uniquely represented as $d=y+x$, $y \in C$, $x \in C$, is that $d=\delta_1 1\delta_2 1\delta_3 1\delta_4 1\ldots$adinf., where no δ_{2k-1} should contain a2 and no δ_{2k} should contain a 0.

Note 2. It follows that $d = \frac{1}{2} = (\cdot 111\ldots\text{ad inf})$ is the only point in $0 < d < 1$, which can be uniquely represented both as $y+x = d$, and $y-x = d$, where $x \in C$, $y \in C$.

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ON AXIALLY SYMMETRIC SUPERPOSABLE FLOWS-II

By

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1. Introduction. Axially-symmetric inviscid flows with poloidal components only were considered from the point of view of superposability by Ram Ballabh (1955). Bhatnagar and Varma (1957) considered these for viscous fluids. Lakshman Rao (1957) considered the superposability of axially-symmetric flows having both poloidal and toroidal components. Kapur (1959*a*) studied the problem of decay of vorticity for self-superposable axially-symmetrical flows when the velocities have both poloidal and toroidal components. In two other papers, Kapur (1959*b*, 1960), has extended these studies to axially-symmetric hydromagnetic superposable flows.

In the present paper, we extend the study started in Kapur (1959*a*) to discuss further properties of axially symmetric viscous flows with both poloidal and toroidal components. In particular we discuss the following:

(*i*) Characterisation of steady axially-symmetric Beltrami *viscous* flows.

(*ii*) Characterisation of steady axially-symmetric self-superposable *viscous* flows.

(*iii*) Most general poloidal and toroidal velocity fields superposable on each other.

(*iv*) Most general velocity field superposable on a toroidal field.

(*v*) Possibility of axially symmetric superposable motions of the type curl $\vec{q}_1 = \lambda_2 \vec{q}_2$, curl $\vec{q}_2 = \lambda_1 \vec{q}_1$ with constant values for λ_1 and λ_2.

The first two problems were also discussed by Lakshman Rao, but as we shall see below, his characterisations are valid for non-viscous fluids only.

2. Steady axially symmetric Beltrami flows. Following Chandrasekhar (1956) and Kopur (1959), we let

$$\vec{q} = -\frac{1}{r}\frac{\partial \psi}{\partial z}\vec{i}_r + \frac{\Omega}{r}\vec{i}_\varphi + \frac{1}{r}\frac{\partial \psi}{\partial r}\vec{i}_z \tag{1}$$

where r, φ, z represent cylindrical coordinates and $\vec{i}_r$, $\vec{i}_\varphi$, $\vec{i}_z$ are the corresponding unit vectors. The velocity vector will be purely poloidal if $\Omega = 0$ and purely toroidal if $\psi = 0$. The choice of for form (*i*) ensures that the equation of continuity, *viz.*

$$\operatorname{div} \vec{q} = 0 \tag{2}$$

is automatically satisfied. Again from (1)

$$\operatorname{curl} \vec{q} = -\frac{1}{r}\frac{\partial \Omega}{\partial z}\vec{i}_r - \frac{1}{r}D^2\psi\,\vec{i}_\varphi + \frac{1}{r}\frac{\partial \Omega}{\partial r}\vec{i}_z, \tag{3}$$

where
$$D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{4}$$

For a Beltrami flow
$$\text{curl } \vec{q} = \lambda \vec{q}, \tag{5}$$
where λ is a function x, y, z and possibly also of t. From (1), (3) and (5)
$$\frac{\partial \Omega}{\partial z} = \lambda \frac{\partial \psi}{\partial z} \tag{6}$$
$$\frac{\partial \Omega}{\partial r} = \lambda \frac{\partial \psi}{\partial r} \tag{7}$$
$$D^2\psi = -\lambda \Omega \tag{8}$$

From (6) and (7)
$$\frac{\partial(\Omega, \psi)}{\partial(r, z)} = 0, \tag{9}$$
so that
$$\Omega = f(\psi) \tag{10}$$
and then from (6) or (7)
$$\lambda = f'(\psi), \tag{11}$$
Substituting in (8)
$$D^2\psi + f(\psi)f'(\psi) = 0. \tag{12}$$
(10) and (12) give conditions on $\vec{q}$ so that it may give a Beltrami flow, but these are not enough. In addition, the condition of integrability of the equations must also be satisfied. These are [Kapur (1959), Goldstein (1938)]
$$\frac{\partial}{\partial t} D^2\psi = -\frac{\partial(r^{-2}D^2\psi)}{\partial(z, r)} - \frac{2\Omega}{r^2}\frac{\partial \Omega}{\partial z} + \nu D^4\psi \tag{13}$$
$$r\frac{\partial \Omega}{\partial t} = -\frac{\partial(\Omega, \psi)}{\partial(z, r)} + \nu r D^2\Omega. \tag{14}$$

Using (10) and (12), these reduce to
$$\frac{\partial}{\partial t} D^2\psi = \nu D^4\psi \tag{15}$$
$$\frac{\partial \Omega}{\partial t} = \nu D^2\Omega. \tag{16}$$

If the fluid is inviscid and the flow is steady, (15) and (16) are satisfied so that an axially-symmetric steady Beltrami flow in an inviscid fluid is characterised by (10) and (12) only. Lakshman Rao (1957) deduced these by another method, but it is obvious from the above that his characterisation holds for inviscid fluids only.

If steady axially symmetric Betrami flows are possible in viscuos fluids, then in addition to (10) and (12), the following should also be satisfied.
$$D^2\Omega = 0 \tag{17}$$
$$D^4\psi = 0 \tag{18}$$

From (10), (12), (17) and (18), we see that ψ must satisfy simultaneously the following three equations:

$$D^2[f(\psi)] = 0, \qquad D^4\psi = 0, \qquad D^2\psi + f(\psi)f'(\psi) = 0 \tag{19}$$

Using the last of these equations, the first two become

$$-f(\psi)f'^2(\psi) + f''(\psi)\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = 0 \tag{20}$$

$$-f(\psi)f'(\psi)[f(\psi)f''(\psi) + f'^2(\psi)]$$

$$+f(\psi)f'''(\psi) + 3f'(\psi)f''(\psi)\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = 0 \tag{21}$$

If ψ is not constant (20) and (21) give

$$\frac{ff'^2}{ff'(ff''+f'^2)} = \frac{f''}{ff'''+3f'f''} \tag{22}$$

Therefore either (i) $f = 0$

or (ii) $f' = 0$

or (iii) $ff'f''' + 2f'^2f'' - ff''^2 = 0$ (23)

In the first case $\Omega = 0$ and the velocity is purely poloidal. Also $\lambda = 0$, curl $\vec{q} = 0$, so that the motion is irrotational. This is a degenerate case. It is even otherwise obvious that a purely poloidal velocity cannot be a Beltrami flow, as from (1) and (3), we see that if the velocity vector is purely poloidal, the vorticity vector is purely toroidal, and similarly if the velocity vector is purely toroidal, the verticity voctor would be purely poloidal. *Thus axially symmetric Beltrami flow must have both poloidal and toroidal components.*

In the second case $\lambda = 0$ and again curl $\vec{q} = 0$

Thus the really significant case is the third. Now (23) can be written as

$$\frac{f'''}{f''} + \frac{2f'}{f} - \frac{f''}{f'} = 0$$

Integrating thrice, we get

$$f(\psi) + a\log(f(\psi) - a) = b\psi + c, \tag{24}$$

where a, b, c are arbitrary constants, From (24)

$$f(\psi)f'(\psi) = b(f(\psi) - a)$$

Thus steady axially-symmetric viscous Beltrami flows are characterised by

$$\Omega = f(\psi), \; D^2\psi + b[f(\psi) - a] = 0, \tag{25}$$

where $f(\psi)$ is given by (24)

In particular when $a = 0$, $c = 0$, we get

$$D^2\psi + b^2\psi = 0, \tag{26}$$

which has been solved by Chandrasekhar (1956)

3. Characterisation of steady axially-symmetric self-superposable viscous flows. For self-superposable flows,

$$\text{curl}\ [\vec{q} \times \text{curl}\ \vec{q}] = 0$$

i.e.

$$\vec{q} \times \text{curl}\ \vec{q} = \text{grad}\ \varphi \tag{27}$$

Using (1), (3) and the axisymmetry of the flow, we get

$$\frac{\partial(\Omega, \psi)}{\partial(r, z)} = 0 \tag{28}$$

$$\frac{1}{r^2}\left[\Omega \frac{\partial\Omega}{\partial r} + D^2\psi\frac{\partial\psi}{\partial r}\right] = \frac{\partial\varphi}{\partial r} \tag{29}$$

$$\frac{1}{r^2}\left[\Omega \frac{\partial\Omega}{\partial z} + D^2\psi\frac{\partial\psi}{\partial z}\right] = \frac{\partial\varphi}{\partial z} \tag{30}$$

From (28)

$$\Omega = f(\psi), \tag{31}$$

and then (29), (30) give

$$\frac{1}{r^2}[f(\psi)f'(\psi) + D^2\psi]\frac{\partial\psi}{\partial r} = \frac{\partial\varphi}{\partial r} \tag{32}$$

$$\frac{1}{r^2}[f(\psi)f'(\psi) + D^2\psi]\frac{\partial\psi}{\partial z} = \frac{\partial\varphi}{\partial z} \tag{33}$$

From (32) and (33)

$$\frac{\partial(\psi, \varphi)}{\partial(r, z)} = 0; \quad \varphi = g(\psi) \tag{34}$$

$$D^2\psi + f(\psi)f'(\psi) = r^2 g'(\psi) \tag{35}$$

(31) and (35) were deduced by Lakshmana Rao by a different method, but again it is seen that they characterise steady inviscid flows only. For viscous fluids, (17) and (18) should also be satisfied. Proceeding as before, we get in place of (20), (21)

$$f'(\psi)[-f(\psi)f'(\psi) + r^2 g'(\psi)] + f''(\psi)\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = 0 \tag{36}$$

$$[f(\psi)f''(\psi) + f'^2(\psi)][-f(\psi)f'(\psi) + r^2 g'(\psi)]$$

$$+ [f(\psi)f'''(\psi) + 3f'(\psi)f''(\psi)]\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] \tag{37}$$

From (36) and (37), we again deduce (23) and (24). *Thus axially-symmetric viscous self-superposable flows are characterised by*

$$\left.\begin{aligned} \Omega = f(\psi),\ D^2\psi+f(\psi)f'(\psi) &= r^2g'(\psi), \\ f(\psi)+a\log\left[f(\psi)-a\right] &= b\psi+c \end{aligned}\right\} \tag{38}$$

4. Non-steady viscous Beltrami flows. We conider the simple case when

$$\Omega = f(\psi) = \lambda\psi, \tag{39}$$

where λ is a constant. In this case (15) and (16) give

$$D^2\left[\frac{\partial\psi}{\partial t}-\nu D^2\psi\right] = 0. \tag{40}$$

$$\alpha\left[\frac{\partial\psi}{\partial t}-\nu D^2\psi\right] = 0 \tag{41}$$

Both are satisfied if

$$\frac{\partial\psi}{\partial t}-\nu D^2\psi = 0 \tag{42}$$

Using (12) and (39)

$$D^2\psi+\lambda^2\psi = 0 \tag{42}$$

From (42) and (43)

$$\frac{\partial\psi}{\partial t}+\nu\lambda^2\psi = 0, \tag{44}$$

so that

$$\psi = \psi_0 e^{-\lambda^2\nu t} \tag{45}$$

and

$$\Omega = \alpha\psi_0 e^{-\lambda\nu t}, \tag{46}$$

where ψ_0 satisfies the equation

$$D^2\psi_0+\lambda^2\psi_0 = 0 \tag{47}$$

which has been solved by Chandrasekhar (1956). We next consider the question whether λ occuring in (5) can be a function of t alone. If possible, let

$$\lambda = \lambda(t)$$

(6), (7) and (8) then give

$$\frac{\partial\Omega}{\partial z} = \lambda(t)\frac{\partial\psi}{\partial z} \tag{48}$$

$$\frac{\partial\Omega}{\partial r} = \lambda(t)\frac{\partial\psi}{\partial r} \tag{49}$$

$$D^2\psi = -\lambda(t)\Omega \tag{50}$$

From (48) and (59)

$$\Omega = \lambda(t)\psi+\mu(t) \tag{51}$$

Substituting in (50)

$$D^2\psi = -\lambda^2(t)\psi-\lambda(t)\mu(t) \tag{52}$$

From (15) and (16), we then get

$$-2\lambda(t)\lambda'(t)\psi-\lambda^2(t)\partial\psi/\partial t-\lambda(t)\mu'(t)-\lambda'(t)\mu(t)$$
$$=\nu D^2[-\lambda^2(t)\psi-\lambda(t)\mu(t)]$$
$$=\nu\lambda^3(t)[\lambda(t)\psi+\mu(t)] \tag{53}$$

and

$$\lambda'(t)\psi+\lambda(t)\partial\psi/\partial t+\mu'(t)$$
$$=-\nu\lambda^2(t)[\lambda(t)\psi+\mu(t)] \tag{54}$$

Multiplying (54) by $\lambda(t)$ and adding to (53), we get

$$\lambda'(t)[\lambda(t)\psi+\mu(t)] = 0$$

If $\lambda'(t) = 0, \quad \lambda(t) = \text{constant.}$

If $\lambda(t)\psi+\mu(t) = 0$, ψ becomes a functiun of t alone, then from (10) Ω becomes a function of t alone and from (13) curl $\vec{q} = 0$, which gives us the degenerate case. Thus we fiind that λ in (5) cannot be a function of t alone. This result is true for all Beltrami flows and was proved by Ram Ballabh (1940). The above independent proof of the result for axially-symmetric flows is however interesting.

5. Steady poloidal and toroidal velocity fields superposable on each other. Let

$$\vec{q}_1 = -\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r+0\,\vec{i}_\varphi+\frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z \tag{55}$$

and

$$\vec{q}_2 = \frac{\Omega}{r}\vec{i}_\varphi \tag{56}$$

be superposable on each other. The necessary sufficient condition for this is

$$\text{curl}\left[\left(-\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r+\frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z\right)\times\left(-\frac{1}{r}\frac{\partial\Omega}{\partial z}\vec{i}_r+\frac{1}{r}\frac{\partial\Omega}{\partial r}\vec{i}_z\right)+\frac{\Omega}{r}\vec{i}_\varphi\times\left(-\frac{1}{r}D^2\psi\,\vec{i}_\varphi\right)\right] = 0.$$

or

$$\text{curl}\left[\frac{1}{r^2}\frac{\partial(\psi,\Omega)}{\partial(r,z)}\vec{i}_\varphi\right] = 0$$

i.e.

$$\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial(\psi,\Omega)}{\partial(r,z)}\right] = 0, \qquad \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(\psi,\Omega)}{\partial(r,z)}\right] = 0, \tag{57}$$

so that

$$\frac{\partial(\psi,\Omega)}{\partial(r,z)} = Ar, \tag{58}$$

where A is a constant.

The conditions of integrability from (55), (56) as obtained from (13) and (14) are:

$$-\frac{\partial[(1/r^2)D^2\psi,\psi]}{\partial(z,r)}+\nu D^4\psi = 0 \tag{59}$$

$$D^2\Omega = 0 \tag{60}$$

$$\partial\Omega/\partial z = 0 \tag{61}$$

From (4), (60) and (61)

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\Omega=0 \qquad (62)$$

or

$$\Omega = Br^2+C.$$

(58) and (61) give

$$-\frac{\partial\psi}{\partial z}\,\frac{\partial\Omega}{\partial r}=Ar.$$

Using (62)

$$\frac{\partial\psi}{\partial z}=-\frac{A}{2B}$$

therefore

$$\psi=-\frac{A}{2B}z+\varphi(r) \qquad (63)$$

Substituting in (59), we get

$$\nu[r^4\varphi''''-2r^3\varphi'''+3r^2\varphi''-3r\varphi']=\frac{A}{2B}[r^3\varphi'''-3r^2\varphi''+3r\varphi'] \qquad (64)$$

Integrating

$$\varphi(r)=\alpha+\beta r^2+\gamma r^4+\delta r^{2+A/(2B\nu)} \qquad (65)$$

so that

$$\vec{q}_1=\frac{1}{r}\,\frac{A}{2B}\,\vec{i}_r+\left[2\beta+4\gamma r^2+\left(2+\frac{A}{2B\nu}\right)\delta r^{A/(2B\nu)}\right]\vec{i}_z \qquad (66)$$

$$\vec{q}_2=\left(Br+\frac{C}{r}\right)\vec{i}_\varphi \qquad (67)$$

Since we require the modified pressure also to be axially symmetric, this would require that $A=0$, so that the most *general poloidal and toroidal viscous velocity fields superposable on each other are given by*

$$\vec{q}_1=[D+Er^2+Fr^2\log r]\vec{i}_z \qquad (68)$$

$$\vec{q}_1=\left[Br+\frac{C}{r}\right]\vec{i}_\varphi \qquad (69)$$

Thus the poloidal field is purely axial and the toroidal field includes the field due to a rectilinear vortex filament of strength $2\pi C$ along the axis.

6. The most general steady axi-symmetric motion superposable on toroidal velocity field. Let

$$\vec{q}_1 = -\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r + \frac{\Omega}{r}\vec{i}_\varphi + \frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z \tag{70}$$

represent the motion superposable on the purely toroidal velocity field

$$\vec{q}_2 = \frac{\bar{\Omega}}{r}\vec{i}_\varphi \tag{71}$$

The conditions of superposability are

$$\frac{\partial}{\partial z}\left[\frac{1}{r^2}\frac{\partial(\Omega\bar{\Omega})}{\partial r}\right] = \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial(\Omega\bar{\Omega})}{\partial z}\right] \tag{72}$$

$$\frac{\partial(\psi, \bar{\Omega})}{\partial(r, z)} = 0 \tag{73}$$

For steady flows, the conditions of integrability give

$$\frac{2\Omega}{r^2}\frac{\partial\Omega}{\partial z} + \frac{1}{r}\frac{\partial(\psi, D^2\psi)}{\partial(r, z)} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}D^2\psi = \nu D^4\psi \tag{74}$$

$$\frac{1}{r}\frac{\partial(\psi_1, \Omega)}{\partial(r, z)} = \nu D^2\Omega \tag{75}$$

$$\frac{2\Omega}{r^2}\frac{\partial\bar{\Omega}}{\partial z} = 0 \tag{76}$$

$$D^2\bar{\Omega} = 0 \tag{77}$$

From (76) and (77)

$$\bar{\Omega} = Br^2 + C \tag{78}$$

From (73) and (76)

$$\psi = -\frac{A}{2B}z + \varphi(r) \tag{79}$$

From (72)

$$\frac{\partial(\Omega\bar{\Omega})}{\partial z} = 0 \tag{81}$$

From (76) and (80)

$$\frac{\partial\Omega}{\partial z} = 0 \tag{81}$$

From (74), 79) and (81), we get (64) for determining $\varphi(r)$. For determining Ω, we use (75) which gives

$$\frac{1}{r}\,\frac{A}{2B}\,\frac{\partial\Omega}{\partial r}=\nu\left(\frac{\partial^2\Omega}{\partial r^2}-\frac{1}{r}\,\frac{\partial\Omega}{\partial r}\right)$$

Giving $\Omega = hr^{2-\frac{1}{2}A/B\nu}+M$

Since $A = 0$, we get

$$\psi = \alpha r^2+Br^2+\gamma r^4+\varepsilon r^2\log r \tag{82}$$

$$\Omega = hr^2+M \tag{83}$$

Thus a purely toroidal steady velocity field in a viscous fluid is given by a vector or the type (69) and the most general steady axi-symmetric motion superposable on it is given by a velocity vector of the type:

$$\vec{q}=[D+Er^2+Fr^2\log r\ ]\,\vec{i_z}+\left(hr+\frac{M}{r}\right)\vec{i_\varphi} \tag{84}$$

Comparison with (68) shows that a purely toroidal steady flow is superposable on itself. Alternatively we could have first proved this result and used it in conjunction with the result of the last section to prove the result of this section.

For the superposation of the general axi-symmetric flows the conditions of superposability are:—

$$\frac{\partial(\psi_1,\Omega_2)}{\partial(z,r)}+\frac{\partial(\psi_2,\Omega_1)}{\partial(z,r)}=0 \tag{85}$$

$$\frac{\partial}{\partial z}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(\Omega_1\Omega_2)+\frac{1}{r^2}\,\frac{\partial\psi_1}{\partial r}\,D^2\psi_2+\frac{1}{r^2}\,\frac{\partial\psi_2}{\partial r}\,D^2\psi_1\right\}$$

$$=\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\,\frac{\partial}{\partial z}(\Omega_1\Omega_2)+\frac{\partial\psi_1}{\partial z}\,D^2\psi_2+\frac{\partial\psi^2}{\partial z}\,D^2\psi_1\right\} \tag{86}$$

and the conditions of integrability are:

$$\frac{2\Omega_1}{r^2}\,\frac{\partial\Omega_1}{\partial z}+\frac{1}{r}\,\frac{\partial(\psi_1,D^2\psi_1)}{\partial(r,z)}+\frac{2}{r^2}\,\frac{\partial\psi_1}{\partial z}\,D^2\psi_1=\nu D^4\psi_1 \tag{87}$$

$$\frac{1}{r}\,\frac{\partial(\psi_1,\Omega)}{\partial(r,z)}=\nu D^2\Omega_1 \tag{88}$$

$$\frac{2\Omega_2}{r^2}\,\frac{\partial\Omega_2}{\partial z}+\frac{1}{r}\,\frac{\partial(\psi_2,D^2\psi_2)}{\partial(r,z)}+\frac{2}{r^2}D^2\psi_2=\nu D^4\psi_2 \tag{89}$$

$$\frac{1}{r}\,\frac{\partial(\psi_2,\Omega_2)}{\partial(r,z)}=\nu D^2\Omega_2 \tag{90}$$

We have studied above the cases when (*i*) $\Omega_1 = 0$, $\psi_2 = 0$ (*ii*) $\psi_2 = 0$. The case when (*iii*) $\Omega_1 = 0$, $\Omega_2 = 0$ was studied earlier by Bhatnagar and Varma (1957) and Kapur (1959*a*). The case (*iv*) $\psi_1 = 0$, $\psi_2 = 0$ is included in our discussion of case (*ii*). We can disuss other cases in the same way as above.

7. Axisymmetric flows of the type curl $\vec{q}_1 = \lambda_2 \vec{q}_2$, curl $\vec{q}_2 = \lambda_1 \vec{q}_1$. Flows of the type have been considered by Ram Ballabh (1943) C. D. Ghildyal (1957) and Devi Singh (1959). We consider the question whether steady flows of this type with constant values of λ_1, λ_2 are possible in viscous fluids.

Using (1) and (3), we get

$$\frac{\partial \Omega_1}{\partial z} = \lambda_2 \frac{\partial \psi_2}{\partial z}, \quad \frac{\partial \Omega_1}{\partial r} = \lambda_2 \frac{\partial \psi_2}{\partial r}, \quad D^2\psi_1 = -\lambda_2 \Omega_2 \tag{91}$$

$$\frac{\partial \Omega_2}{\partial z} = \lambda_1 \frac{\partial \psi_1}{\partial z}, \quad \frac{\partial \Omega_2}{\partial r} = \lambda_1 \frac{\partial \psi_1}{\partial r}, \quad D^2\psi_2 = -\lambda_1 \Omega_1 \tag{92}$$

so that

$$\Omega_1 = \lambda_2 \psi_2, \qquad \Omega_2 = \lambda_1 \psi_1 \tag{93}$$

$$D^2\psi_1 = -\lambda_1\lambda_2\psi_1; \quad D^2\psi_2 = -\lambda_1\lambda_2\psi_2 \tag{94}$$

$$D^4\psi_1 = \lambda_1^2\lambda_2^2\psi_1; \qquad D^4\psi_2 = \lambda_1^2\lambda_2^2\psi \tag{95}$$

Using these, the conditions of integrability become:

$$\frac{2}{r^2}\left(\lambda_2\psi_2 \frac{\partial \psi_2}{\partial z} - \lambda_1\psi_1 \frac{\partial \psi_1}{\partial z}\right) = \nu\lambda_1^2\lambda_2\psi_1 \tag{96}$$

$$\frac{1}{r}\frac{\partial(\psi_1, \psi_2)}{\partial(r, z)} = -\nu\lambda_1\lambda_2\psi_2 \tag{97}$$

$$\frac{2}{r^2}\left[\lambda_1\psi_1 \frac{\partial \psi_1}{\partial z} - \lambda_2\psi_2 \frac{\partial \psi_2}{\partial z}\right] = \nu\lambda_2\lambda_1\psi_2 \tag{98}$$

$$\frac{1}{r}\frac{\partial(\psi_2, \psi_1)}{\partial(r, z)} = -\nu\lambda_1\lambda_2\psi_1 \tag{99}$$

From (97) and (99)

$$\psi_1 = -\psi_2 = 0$$

and then fron (98)

$$\Omega_1 = \Omega_2 = 0$$

Thus we deduce that steady axially-symmetric superposable flows of the type curl $\vec{q}_1 = \lambda_2 \vec{q}_2$, *curl* $\vec{q}_2 = \lambda_2 \vec{q}_1$ *are not possible in visous fluids, if* λ_1 *and* λ_2 *are constant,*

For ideal fluids, the above equations give

$$\frac{\partial(\psi_1, \psi_2)}{\partial(r, z)} = 0; \quad \lambda_1\psi_1 \frac{\partial \psi_1}{\partial z} - \lambda_2\psi_2 \frac{\partial \psi_2}{\partial z} = 0 \tag{100}$$

These give

$$\lambda_1\psi_1 \frac{\partial \psi_1}{\partial r} - \lambda_2\psi_1 \frac{\partial \psi_2}{\partial r} = 0 \tag{101}$$

From (100) $\lambda_1\psi_1^2-\lambda_2\psi_2^2$ can be a function of r only and from (101) it can be a function of z only.

Therefore
$$\lambda_1\psi_1^2-\lambda_2\psi_2^2 = \text{const.} \tag{102}$$

The constant can be chosen to be zero, then

$$\psi_1 = \psi,\ \psi_2 = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}}\psi,\ \Omega_2 = \lambda_1\psi,\ \Omega_1 = (\lambda_1\lambda_2)^{\frac{1}{2}}\psi \tag{103}$$

where ψ is any arbitrary function of r and z.

Thus steady axially-symmetric superposable flows of the type curl $\vec{q}_1 = \lambda_2\vec{q}_2$, *curl* $\vec{q}_2 = \lambda_1\vec{q}_1$ *are possible in inviscid flows and are given by.*

$$\vec{q}_1 = -\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r + (\lambda_1\lambda_2)^{\frac{1}{2}}\frac{\psi}{r}\vec{i}_\theta - \frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z \tag{104}$$

$$\vec{q}_2 = -\frac{1}{r}\frac{\partial\psi}{\partial z}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}}\vec{i}_r + \lambda_1\frac{\psi}{r}\vec{i}_\theta + \frac{1}{r}\frac{\partial\psi}{\partial r}\left(\frac{\lambda_1}{\lambda_2}\right)\vec{i}_z \tag{105}$$

Summary. In the present paper we have obtained *the most general* steady axially-symmetric flows of the following types:

(*i*) Beltrami Flows in viscous fluids.

(*ii*) Self superposable flows in viscous fluids.

(*iii*) Flows of the type curl $\vec{q}_1 = \lambda_2\vec{q}_2$, curl $\vec{q}_2 = \lambda_1\vec{q}_1$ (with constant values for λ_1, λ_2) in inviscid fluids (For viscous fluids it has been shown that such flows do not exist).

(*iv*) Poloidal and toroidal flows superposable on each other.

(*v*) Flows superposable on a purely toroidal flow.

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ON GENERALIZED HERMITE POLYNOMIALS

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1. Introduction. The classical orthogonal polynomials have been generalized in different direction. For the basis of these generalisations in some cases Rodrigue's formula is taken, in others generating function, differential equation, recurrence and so on.

Let $k \geq 2$ be a fixed natural number. We denete by $D_k^{(mk)}$ the result of m—times iteration of the operator $D_k^{(k)} \equiv \frac{d}{dz}\frac{1}{z^{k-2}}\frac{d}{dz}$. Further on, if $D_k^{(k-1)} \equiv \frac{1}{z^{k-3}}\frac{d}{dz}$, then $D_k^{(mk+k-1)} \equiv \frac{1}{z^{k-2}}\frac{d}{dz}D_k^{(mk)}$. It is clear that when $k = 2$, $D_2^{(2m)} \equiv \frac{d^{2m}}{dz^{2m}}$ and $D_2^{(2m+1)} \equiv \frac{d^{2m+1}}{dz^{2m+1}}$. The differential operators $D_k^{(mk)}$ and $D_k^{(mk+k-1)}$ generate the Rodrigue's type formulas. As for example. the generalized Legendre polynomials introduced by the author of the present paper is defined by the following formulas (Kharadzé, 1935)

$$Q_{(mk)}(z) \equiv \frac{1}{k^{2m}(2m)!} D_k^{(mk)} (z^k-1)^{2m},$$

$$Q_{(mk+1)}(z) \equiv \frac{1}{k^{2m+1}(2m+1)!} D_k^{(mk+k-1)} (z^k-1)^{2m+1}.$$

A. Sharma (1948) has studied these polynomials in detail. P. Chatterjee (1955) hss considered the generalized Hermite polynomials expressed by the Rodrigue's type formulas with the cperators $D_k^{(mk)}$ and $D_k^{(mk+k-1)}$ as well[1]:

$$H_{(mk)}(z) \equiv e^{z^k/k} D_k^{(mk)}(e^{-z^k/k})\,;\; H_{(mk)}(z) \equiv -e^{z^k/k} D_k^{(mk+k-1)}(e^{-z^k/k}). \tag{1}$$

In particular, when $m = 0, 1, 2,$ we have

$$H_{(0)}(z) \equiv 1, \quad H_{(k)}(z) \equiv z^k - 1, \quad H_{(2k+1)}(z) \equiv z^{2k} - 2(k+1)z^k + k + 1,$$

$$H_{(1)}(z) \equiv z, \quad H_{(k+1)}(z) \equiv z^{k+1} - (k+1)z. \quad H_{(2k+1)}(z) \equiv z^{2k+1} - 2(2k+1)z^{k+1} + (k+1)(2k+1)z.$$

The polynomial $H_{(mk)}(z)$ contains only the powers of species z^{pk} and $H_{(mk+1)}(z)$—only the powers z^{pk+1}, $(p = 0, 1, \ldots m)$.

As it was shown by P. Chatterjee, (1955) the polynomials $\{H_{(mk)}(z)\}$, $\{H_{(mk+1)}(z)\}$ satisfy the following relations of orthogonality

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(1) We write the exponential function in the form $e^{-z^k/k}$ while P. Chatterjee prefers to write e^{-z^k}. As it was indicated by A. Erdélyi (*Math. Reviews*, 1956. p. 967) the polynomials $\{H_{(mk)}\}$ and $\{H_{(mk+1)}\}$ are expressed by the Laguerre polynomials

$$\int_0^\infty e^{-z^k/k} H_{(mk)}(z) H_{(nk)}(z)\,dz = 0, \quad m - n,$$

$$\int_0^\infty z^{k-2} e^{-z^k/k} H_{(mk)+1}(z) H_{(nk+1)}(z)\,dz = 0, \quad m \neq n. \tag{2}$$

For these polynomials the following identities are true

$$H'_{(mk+1)}(z) = (mk+1)H_{(mk)}(z),\ H'_{(mk)}(z) = mkz^{k-2}H_{((m-1)k+1)}(z). \tag{3}$$

In the present paper some new facts are stated conberning these polynomials. In n°2 a generalized exponential function expressed by the Bessel functions is introduced, by means of which a new generating function for the polynomials $\{H_{(mk)}(z)\}$ and $\{H_{(mk+1)}(z)\}$ is constructed. n°3 is devoted to the application of Turán's method for the definition of the domain of zeros location for the polynomial $f(z)$ representable as a linear combination of generalized Hermite polynomials.

2. The generalized exponential function $E_k^{(z)}$ and the generating function for the polynomials $\{H_{(mk)}(z)\}$ and $\{H_{(mk+1)}(z)\}$. In the above cited article, P. Chatterjee suggests two different generating functions for each sequence $\{H_{(mk)}(z)\}$ and $\{H_{(mk+1)}(z)\}$ separately. These functions are expressed by means of an ordinary exponential function. In the present n° we shall show that introducing special integral function $E_k(z)$ – generalized exponential – we can represent the generating function as $e^{-t^k/k}E_k(zt)$, good at once for all the polynomials of both sequences.

Let $k \geq 2$ be a fixed natural number as it was above. We form the sequence (s_k) of integers of the species mk and $mk+1$, $(m = 0, 1, 2 \ldots)$:

(s_k) $\qquad 0, 1, k, k+1, 2k, 2k+1, \ldots, mk, mk+1, \ldots$

The product of these numbers from unit to some mk or $mk+1$ presents the generalized factorial, associated with the sequence (s_k) we denote it by $\{mk\}!$ or $\{mk+1\}!$ respectively. When $k = 2$, we have, obviously, the ordinary factorial associated with natural sequence. By the same sequence (s_k) we form the generalized binomial coefficients $\begin{bmatrix} mk \\ p \end{bmatrix}$, $p \leq 2m$ or $\begin{bmatrix} mk+1 \\ q \end{bmatrix}$, $q \leq 2m+1$, according to the following rule: for arbitrary mk and given $p \leq 2m$ we form the fraction, the numerator of which presents the product of p factors from the sequence (s_k), beginning with mk in a decerasing order and the denominator is the generalized factorial consisting of p factors. The similar fraction can be constructed for $mk+1$ and given $q \leq 2m+1$. The two following examples show the structure of the generalized binomial coefficients:

$$\begin{bmatrix} 2k \\ 3 \end{bmatrix} = \frac{2k(k+1)k}{1.k(k+1)} = \frac{2k(k+1)k}{\{k+1\}!} = 2k,$$

$$\begin{bmatrix} 3k+1 \\ 4 \end{bmatrix} = \frac{(3k+1)3k(2k+1)3k}{1.k(k+1)2k} = \frac{(3k+1)3k(2k+1)2k}{\{2k\}!} = \frac{3(3k+1)(2k+1)}{k+1}.$$

Further-on, we introduce the generalized binomial $[a+b]^{mk}$ or $[a+b]^{mk+1}$, associated with the sequence (s_k); this is the polynomial in a and b, the algebraic structure of which is analogous to an ordinary binomial. Then, $[a+b]^{mk+1}$ represents the sum $\Sigma M_t a^r b^s$

where the exponent r takes the value from the sequence (s_k) beginning from $mk+1$ to zero, and s increases from zero to $mk+1$, taking the value from the same sequence (s_k). As to the numerical factors M_i, they present the corresponding binomial coefficients

$$\begin{bmatrix} mk+1 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} mk+1 \\ 1 \end{bmatrix}, \begin{bmatrix} mk+1 \\ 2 \end{bmatrix}, \ldots, \begin{bmatrix} mk+1 \\ 2m+1 \end{bmatrix} = 1.$$

For example $$[a+b]^{2k+1} = a^{2k+1} + \frac{2k+1}{1} a^{2k}b + \frac{(2k+1)2k}{1.k} a^{k+1}b^{k}$$

$$+ \frac{(2k+1)2k(k+1)}{1\ k(k+1)} a^{k}b^{k+1} + \frac{(2k+1)2k(k+1)k}{1.k(k+1)2k}\ ab^{2k} + \frac{(2k+1)2k(k+1)k.1}{1.k(k+1,2k(2k+1)} b^{2k+1}.$$

When $k=2$ we have the ordinary binomial $(a+b)^5$.

Let us consider now the integral function $E_k(z)$, the Taylor-expansion of which is

$$E_k(z) = 1 + \frac{z}{\{1\}!} + \frac{z^k}{\{k\}!} + \frac{z^{k+1}}{\{k+1\}!} + \ldots + \frac{z^{mk}}{\{mk\}!} + \frac{z^{mk+1}}{\{mk+1\}!} + \ldots =$$

$$= 1 + \frac{z}{1} + \frac{z^k}{1.k} + \frac{z^{k+1}}{1.k(k+1)} + \frac{z^{2k}}{1.k(k+1)2k} + \ldots \qquad (4,$$

It is clear that $E_2(z) = e^z$.

It will be shewn below that

$$e^{-t^k/k}E_k(zt) = H_{(0)}(z) + \frac{t}{1} H_{(1)}(z) + \frac{t^k}{1.k} H_{(k)}(z) + \frac{t^{k+1}}{1.k(k+1)} H_{(k+1)}(z) + \frac{t^{2k}}{1\ k(k+1)2k} H_{(2k)}(z) + \ldots (5)$$

i.e. the left-hand side of this equality presents the generating function of both sequences of the polynomials $\{H_{(mk)}(z)\}$ and $\{H_{(mk+1)}(z)\}$. It follows that the function $E_k(z)$ is most naturally associated with the generalized hermitian polynomials. When $k=2$, we have $e^{-t^2/2}e^{zt}$ — the generating function for the classical polynomials.

As it is seen from (4), this series consists of the two parts: the first component denoted by $\mathrm{ch}_k(z)$ contains only the powers of species z^{pk} and the second $\mathrm{sh}_k(z)$ — only the powers of species z^{pk+1}.

Thus $$E_k(z) = \mathrm{ch}_k(z) + \mathrm{sh}_k(z),$$

where $$\mathrm{ch}_k(z) = 1 + \frac{z^k}{1.k} + \frac{z^{2k}}{1\ k(k+1)2k} + \ldots$$

$$\mathrm{sh}_k(z) = \frac{z}{1} + \frac{z^{k+1}}{1.k(k+1)} + \frac{z^{2k+1}}{1.k(k+1)2k(2k+1)} + \ldots$$

The functions $\mathrm{ch}_k(z)$ and $\mathrm{sh}_k(z)$ are analogous to $\mathrm{ch}z$ and $\mathrm{sh}z$. We immediately obtain

$$\mathrm{sh}'_k(z) = \mathrm{ch}_k(z), \qquad \mathrm{ch}'_k(z) = z^{k-2}\mathrm{sh}_k(z)$$

and, consequently, the function $\mathrm{sh}_k(z)$ satisfies the differential equation

$$w'' - z^{k-2}w = 0$$

and the function $\mathrm{ch}_k(z)$ — the equation

$$zw'' - (k-2)w - z^{k-1}w = 0.$$

Since, $\mathrm{sh}_k(0) = 0, \quad \mathrm{sh}'_k(0) = 1,, \quad \mathrm{ch}_k(0) = 1, \quad \mathrm{ch}'_k(0) = 0,$

we can express the functions $\mathrm{ch}_k(z)$ and $\mathrm{sh}_k(z)$ by means of the Bessel function as follows

$$\mathrm{ch}_k(z) = \left(\frac{2i}{k}\right)^{(k-1/k)} \Gamma\left(\frac{1}{k}\right) z^{k-1/2} J_{-(k-1)/k}\left(\frac{2i}{k} z^{k/2}\right),$$

$$\mathrm{sh}_k(z) = \left(\frac{k}{2i}\right)^{1/k} \Gamma\left(\frac{k+1}{k}\right) z^{\frac{1}{2}} J_{1/k}\left(\frac{2i}{k} z^{k/2}\right).$$

Before to prove the formula (5), we shall make some more preliminary remarks. Let $\{\varphi(z)\}$ be the class of functions representable with power series of the following species:

$$(S_k) \qquad \varphi(z) = a_0 + a_1 z + a_k z^k + a_{k+1} z^{k+1} + a_{2k} z^{2k} + a_{2k+1} z^{2k+1} + \dots$$

We call A_k^h — *transformation of* (S_k) the series, received from the (S_k) by substitution of each power z^{pk} or z^{pk+1} by the corresponding generalized binomial $[z+h]^{pk}$ or $[z+h]^{pk+1}$. We denote the result of this substitution by $\varphi[z; h]$.

Thus, $\varphi[z; h] = a_0 + a_1[z+h]^1 + a_k[z+h]^k + a_{k+1}[z+h]^{k+1} + \dots$

When $k = 2$, it is obvious that the A_2^h — transformation is reduced to translation, namely,

$$\varphi(z+h) = a_0 + a_1(z+h) + a_2(z+h)^2 + a_3(z+h)^3 + \dots$$

One may represent the seriees (S_k) as the sum of two series

$$\varphi(z) = \varphi_{(mk)}(z) + \varphi_{(mk+1)}(z),$$

where the first component $\varphi_{(mk)}(z)$ contains only the powers of species z^{pk} — this is the component of type I, and the second $\varphi_{(mk+1)}(z)$ containing the powers of species z^{pk+1} we call the component of type II.

It is easy to show that

$$\lim_{h \to 0} \frac{\varphi[z; h] - \varphi(z)}{h} = \frac{\varphi'_{(mk)}(z)}{z^{k-2}} + \varphi'_{(mk+1)}(z). \tag{7}$$

Obviously, when $k = 2$ we have the ordinary derivative $\varphi'(z)$. In order to establish the equality (7) it is sufficient to arrange the right-hand side of (6) by the powers of h as follows

$$\varphi[z; h] = A_0 + \frac{h}{1} A_1 + \frac{h^k}{1.k} A_k + \frac{h^{k+1}}{1.k(k+1)} A_{k+1} + \dots \tag{8}$$

From here it is not difficult to calculate that

$$A_0 = \varphi(z) = \varphi_{(mk)}(z) + \varphi_{(mk_1+)}(z)\,; \; A_1 = \frac{\varphi'_{(mk)}(z)}{z^{k-2}} + \varphi'_{(mk+1)}(z)\,;$$

$$A_2 = \left(\frac{\varphi'_{(mk)}(z)}{z^{k-2}}\right)' + \frac{\varphi''_{(mk+1)}(z)}{z^{k-2}};\ldots$$

We introduce the denotation

$$d_k\varphi = \frac{1}{z^{k-2}}\frac{d}{dz}\varphi_{(mk)}(z) + \frac{d}{dz}\varphi_{(mk+1)}(z).$$

Thus, if $\varphi(z)$ is the sum of two components $\varphi_{(mk)}(z)$ and $\varphi_{(mk+1)}(z)$, then the application of operator d_k to function $\varphi(z)$ denotes, that the operator $(1/z^{k-2})\,d/dz$ acts upon the component of type I and the operator d/dz – upon the component of type II. Here we note that applying the operator $(1/z^{k-2})\,d/dz$ to component of type I, we have the series consisting from the powers of species z^{pk+1} and, consquently, the component changes its type, *i.e.* it is transformed into the component of type II. Similarly the application of the operator d/dz to second component changes its type. Hence, repeating the operation d_k, we have

$$d_k^2\varphi \equiv d_k(d_k\varphi) = \frac{d}{dz}\left(\frac{1}{z^{k-2}}\frac{d}{dz}\varphi_{(mk)}\right) + \frac{1}{z^{k-1}}\,\frac{d}{d}\left(\frac{d}{dz}\varphi_{(mk+1)}\right),$$

where the first component is again of type I and the second – of type II.

Proceeding the application of the operator d_k every time the derivative from the component of type I is divided into z^{k-2} and from the second component it is taken simply the derivative.

Thus, if $d_k^n\varphi = d_k(d_k^{n-1}\varphi)$ then we can rewrite the equality (8) as follows

$$\varphi[z;\, h] = \varphi(z) + \frac{h}{1}\,d_k\varphi(z) + \frac{h^k}{1.k}\,d_k^2\varphi(z) + \frac{h^{k+1}}{1.k(k+1)}d_k^3\varphi(z) + \ldots \tag{9}$$

Thus is analogous to Taylor's formula. From here the equality (7) follows immediately.

Let us now return to the function $E_k(z)$. As it is seen from the construction of the series (4),

$$d_kE_k(z) = E_k(z)$$

and, consequently, applying the formula (9) to the function $E_k(z)$, we have

$$E_k[z\,;\,h] = 1 + \frac{[z+h]^1}{1} + \frac{[z+h]^k}{1.k} + \frac{[z+h]^{k+1}}{1.k(k+1)} + \ldots = \tag{10}$$

$$= E_k(z)\left(1 + \frac{h}{1} + \frac{h^k}{1.k} + \frac{h^{k+1}}{1.k(k+1)} + \ldots\right) = E_k(z)E_k(h).$$

In other words, the product $E_k(z)E_k(h)$ gives the A_k^h – transformation of the series $E_k(z) = 1 + \frac{z}{1} + \frac{z^k}{1.k} + \frac{z^{k+1}}{1.k(k+1)} + \ldots$. It is clear that when $k = 2$ the equality (10) takes the following form

$$e^{z+h} = e^z e^h,$$

as, when $k = 2$, the polynomials $[z+h]^{mk}$ and $[z+h]^{mk+1}$ degenerate into ordinary bionomials and, consequently, the A_z^h- transformation of the exponential series is reduced to the change of z by the sum $z+h$.

Now it is not difficult to establish the justification of the expansion (5). Indeed, it is easy to see that the left-hand side of the equality (5), as the function of t is developed into power series of the species

$$e^{-t^k/k}E_k(zt) = Q_0(z) + tQ_1(z) + t^kQ_k(z) + t^{k+1}Q_{k+1}(z) + \dots \tag{11}$$

where the coefficients $\{Q_{pk}\}$ and $\{Q_{pk+1}\}$ represent the polynomials of z; at the same time $Q_{pk}(z)$ contains only the powers of species z^{nk} and $Q_{pk+1}(z)-$ the powers z^{mk+1}. Thus, the expansion of the function $e^{-t^k/k}E_k(zt)$ consists of the two components, one of which is of type I and the second of type II concerning both variables z and t.

Applying the operator d_k to the equality (11), where z is considered as variable and with respect to the invariability of the function $E_k(z)$ concerning this operation, we have

$$te^{-t^k/k}E_k(zt) = \frac{Q_0'(z)}{(zt)^{k-2}} + tQ_1'(z) + t^k\frac{Q_k'(z)}{(zt)^{k-2}} + t^{k+1}Q'_{k+1}(z) + \dots$$

or otherwise

$$t^{k-1}e^{-t^k/k}E_k(zt) = \frac{Q_0'(z)}{z^{k-2}} + t^{k-1}Q_1'(z) + t^k\frac{Q_k'(z)}{z^{k-2}} + t^{2k-1}Q'_{k+1}(z) + \dots$$

Comparing this equality with (11), we have

$$Q_0'(z) = 0,\ Q_1'(z) = Q_0(z),\ Q_k'(z) = z^{k-2}Q_1(z),\ Q'_{k+1}(z) = Q_k(z),$$

and in general

$$Q'_{mk}(z) = z^{k-2}Q_{(m-1)k+1}(z); \quad Q'_{mk+1}(z) = Q_{mk}(z).$$

Returning to the expansion (5), we see, that

$$H_{(1)}'(z) = H_{(0)}(z),\ H_{(k)}'(z) = kz^{k-2}H_{(1)}(z),\ H'_{(k+1)}(z) = (k+1)H_{(k)}(z), \dots$$
$$\dots,\ H'_{(mk)}(z) = mkz^{k-2}H_{((m-1)k+1)}(z), \quad H'_{(mk+1)}(z) = (mk+1)H_{mk}(z),$$

Since it is obvious that $H_{(0)}(z) \equiv 1$, $H_{(1)}(z) \equiv z$, in virtue of the equalities (3), we cunclude, that the cofficients $\{H_{(mk)}(z)\}$ and $\{H_{(mk+1)}(z)\}$ of the expansion (5) present in fact the generalized hermitian polynomials, defined by formulas (1).

Let us note here an interesting expansion of the function $E_k(z)$ by generalized polynomials. Putting in (5) $t = 1$, we have

$$e^{-1/k}E_k(z) = H_{(0)}(z) + \frac{H_{(1)}(z)}{1} + \frac{H_{(k)}(z)}{1.k} + \frac{H_{(k+1)}(z)}{1.k(k+1)} + \dots$$

When $k = 2$ this equality gives the well known expansion by the classical hermitian polynomials:

$$\frac{1}{(e)^{\frac{1}{2}}}e^z = H_0(z) + \frac{H_1(z)}{1} + \frac{H_2(z)}{2!} + \frac{H_3(z)}{3!} + \dots$$

At last we remark, that if $w = e^{\pi i/k}$, *i.e.* $w^k = -1$, then

$$E_k(wz) = c_k(z) + w s_k(z), \tag{12}$$

where

$$C_k(z) = 1 - \frac{z^k}{1.k} + \frac{z^{2k}}{1\ k(k+1)2k} - \dots$$

$$S_k(z) = \frac{z}{1} - \frac{z^{k+1}}{1.k(k+1)} + \frac{z^{2k+1}}{1.k(k+1)2k(2k+1)} - \dots$$

These functions are analogous to $\cos z$ and $\sin z$. When $k = 2$, the equality (12) gives the classical Euler's formula $e^{iz} = \cos z + i \sin z$.

3. **On the location of zeros of some particular classes of polynomials.** If the algebraic polynomial $f(z)$ of degree n is written down in ordinary form, as the sum of powers of variable z

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n, \quad a_n \neq 0$$

then, as it is known, according to the formulas of Cauchy and Walsh, by means of coefficients a_i one may define the circular region including within all the zeros of the polynomial $f(z)$. P. Tura'n paid attention to the problem of the definition of the domain of zeros location depending on the coefficients of the expansion of the given polynomial by some orthogonal system of polynomials. In connection with this problem Tura'n (1954) has considered the case corresponding to classical hermitian polynomials $\{H_p(z)\}$, when the given polynomial $f(z)$ is represented by the expansion

$$f(z) = b_0 H_0(z) + b_1 H_1(z) + \dots + b_n H_n(z), \qquad b_n \neq 0.$$

As it was proved by Tura'n, the stripe including all the zeros of $f(z)$ is defined by inequality

$$|y| \leq \tfrac{1}{2}\left(1 + \frac{M}{|b_n|}\right), \ M = \max |b_i|, \ i = 0, 1, \dots n-1; \ y = y_{mz},$$

which is analogous to Cauchy's inequality[1]. The Tura'n's method is essentially based upon the two properties of ordinary Hermite polynomials: 1) all their zeros are real, 2) the sequence $\{H_n(z)\}$ belongs to the class of Appell's polynomials, *i.e.* $H'_n(z) = nH_{n-1}(z)$.

In this n° we shall show, that, although the generalized polynomials $\{H_{(mk)}(z)\}$, $\{H_{(mk+1)}(z)\}$ do not belong to Appell's sequence and not all their zeros are real, nevertheless one may apply Turán's method to this case as well, and consequently, for those classes of polynomials $f(z)$, which are representable by linear combination of the polynomials $\{H_{(mk)}(z)\}$ or $\{H_{(mk+1)}(z)\}$, one may define the domain of zeros location by the coefficients of linearrepresentation mentioned above.

First of all let us remark that in virtue of the property of orthogonality (2) it is easy to see that all the zeros of the polynomials $H_{(mk)}z)$ and $(H_{(mk+1)}(z)$ are disposed upon the k rays, issuing from the origin and making the angles $2\pi p/k$, $p = 0, 1, \dots, (k-1)$ with the real axis.

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[1] W. Specht (1956) had studied the problem in general case

Now, following Turán's idea we shall show that if the polynomial $f(z)$ of degree $mk+1$ may be represented as the linear combination of generalized Hermite polynomials of species $H_{(pk+1)}(z)$, $p=0,1,\ldots,m$, then by coefficients of this representation one may define the domain of zeros location for $f(z)$.

Let Δ_p denote the total straight line upon which there lies the ray with the inclination $\frac{2\pi}{k}p$.

Let z be any point of a complex plane, not lying upon the lines Δ_p, $p=0, 1,\ldots,(k-1)$. We denote by $\varrho(z, \Delta_p)$ the absolute value of distance from z to the line Δ_p; besides, let

$$h_{(z,k)}=\min \varrho(z, \Delta_p), \quad p=0,1,\ldots,(k-1).$$

We shall prove the following

Theorem. *If the polynomial $f(z)$ of degree $mk+1$ is representable linearly by the generalized Hermite polynomials of species $H_{(nk+1)}(z)$*

$$f(z)=C_0H_{(1)}(z)+C_1H_{(k+1)}(z)+C_2H_{(2k+1)}(z)+\ldots+C_mH_{(mk+1)}(z), \quad C_m\neq 0,$$

then all the zeros of $f(z)$ lie in the domain defined by the inequality

$$h_{(z,k)}\leq \sqrt[k]{1+\frac{M}{|C_m|}}, \tag{13}$$

where $M=\max|C_i|$, $i=0,1,\ldots,(m-1)$.

When $k=2$, it is obvious that $h_{(z,2)}=Y_m(z)$ and, thus in this case the inequality (13) defines the stripe parallel to the real axis; as to the polynomials $\{H_{(2m+1)}(z)\}$, they represent the classical hermitian polynomials of odd degrees. When $k=3$, the star-shaped domain, consisting of three intersecting stripes with common central hexagonal domain corresponds to the inequality (13); these stripes are parallel to three rays the inclinations $0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

The proof of the theorem is based on the following

Lemma. *If z does not lie upon the lines (Δ_p), then the inequality*

$$\left|\frac{H_{(qk+1)}(z)}{H_{(mk+1)}(z)}\right|\leq \frac{1}{h_{(z,k)}k(m-q)} \tag{14}$$

takes place for all the integers q on condition $0\leq q\leq m-1$.

Let $z_{i,mk+1}$, $i=1,\ldots mk+1$, be the zeros of the polynomial $H_{(mk+1)}(z)$. They all lie upon the lines (Δ_p) and, consequently, if z does not lie upon (Δ_p), then $|z-z_{i,mk+1}|\geq h_{(z,k)}$ or otherwise

$$\frac{1}{|z-z_{i,mk+1}|}\leq\frac{1}{h_{(z,k)}}.$$

Since in virtue of (3)

$$\frac{H'_{(mk+1)}(z)}{H_{(mk+1)}(z)} = (mk+1)\frac{H_{(mk)}(z)}{H_{(mk+1)}(z)}$$

hence

$$\left|\frac{H_{(mk)}(z)}{H_{(mk+1)}(z)}\right| = \frac{1}{mk+1}\left|\sum_{i=1}^{mk+1}\frac{1}{z-z_{i,\,mk+1}}\right| \leq \frac{1}{mk+1}\cdot\frac{mk+1}{h_{(z,k)}} = \frac{1}{h_{(z,k)}}. \tag{15}$$

On the other hand, the second of these two equalities (3) leads up to following relation

$$\frac{H'_{(mk)}(z)}{H_{(mk)}(z)} = mkz^{k-2}\frac{H_{(mk-1)k+1}(z)}{H_{(mk)}(z)}.$$

From here we have

$$\left|\frac{H_{(m-1)k+1}(z)}{H_{(mk)}(z)}\right| = \frac{1}{mk\,|z|^{k-2}}\left|\sum_{i=1}^{mk}\frac{1}{z-z_{i,\,mk}}\right| \leq \frac{1}{|z|^{k-2}h_{(z,k)}},$$

where $z_{i,\,mk}$ denote the zeros of the polynomial $H_{(mk)}(z)$.

Since $|z| \geq h_{(z,k)}$ then the latter inequality takes the form

$$\left|\frac{H_{((m-1)k+1)}(z)}{H_{(mk)}(z)}\right| \leq \frac{1}{h^{k-1}_{(z,k)}}, \quad z\bar{\in}(\Delta_p). \tag{16}$$

In virtue of (15) and (16) we receive

$$\left|\frac{H_{((m-1)k+1)}(z)}{H_{(mk+1)}(z)}\right| \leq \frac{1}{h^{k}_{(z,k)}}, \quad z\bar{\in}(\Delta_p).$$

If $q \leq m-1$ then we can write

$$\left|\frac{H_{(qk+1)}(z)}{H_{(mk+1)}(z)}\right| = \prod_{r=q+1}^{m}\left|\frac{H_{((rk-1)k+1)}(z)}{H_{(r+1)}(z)}\right|.$$

Applying the inequality (16) to every factor under the sign of product, we get the required inequality (14).

Now it is not difficult to show that in every domain, where

$$h_{(z,k)} > \sqrt[k]{1+\frac{M}{|C_m|}},$$

we have $|f(z)| > 0$. But this denotes that all the zeros of $f(z)$ lie in the domain, defined by the theorem.

Indeed, from the given representation of $f(z)$ immediately follows that

$$|f(z)| \geq |C_m| \left|H_{(mk+1)}(z)\right| \left\{ 1 - \sum_{q=0}^{m-1} \frac{|C_q|}{|C_m|} \left| \frac{H_{(qk+1)}(z)}{H_{(mk+1)}(z)} \right| \right\},$$

But according to the conditions (14) and (17) we have

$$|f(z)| > |C_m| \left|H_{(mk+1)}(z)\right| \left\{ 1 - \frac{M}{|C_m|} \cdot \frac{1}{h^k_{(z,k)} - 1} \right\}.$$

As z does not lie upon the lines (Δ_p), then $H(z)_{(mk+1)} \neq 0$ and, consequently, if

$$1 - \frac{M}{|C_m|} \cdot \frac{1}{h^k_{(z,1)} - 1} > 0,$$

then $|f(z)| > 0$. This proves the theorem.

According to Tura'n's reasoning by anlogous transformation one may obtain the inequality of Walsh's type, namely, all the zeros of $f(z)$ lie in the domain

$$h_{(z,k)} \leq \sum_{r=0}^{m-1} \sqrt[m-r]{\left|\frac{C_r}{C_m}\right|}.$$

We have considered above the expansion of $f(z)$ by the generalized polynomials of species $H_{(pk+1)}(z)$ The corresponding results could be obtained in case of representaion of $f(z)$ by the polynomials of species $H_{(pk)}(z)$.

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ON A NON-MEASURABLE MAPPING

By

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A one one mapping of a set E onto a set E^* in the same or different spaces is said to be measurable, if the measurability of a subset A of E implies the measurability of its image $A^* \subset E^*$ and vice versa [Carathéodory, 1927, p. 354]. For the measurability of a (1, 1) mapping of E onto E^*, it is necessary that every subset $A(\subset E)$ of measure zero should go to a sub-set $A^*(\subset E^*)$ of measure zero and vice versa. This condition is also sufficient for a continuous (1, 1) mapping. In particular, every linear transformation of non-zero determinant furnishes a measurable mapping. Steadily increasing continuous functions which are absolutely continuous and whose inverses are also absolutely continuous provide simplest examples of such mapping of an interval $a \leq x \leq b$ onto an interval $c \leq y \leq d$ while a steadily increasing continuous function which is not absolutely continuous gives a (1, 1) continuous mapping of a linear interval $a \leq x \leq b$ onto a linear interval $c \leq y \leq d$ which is not measurable according to the definition given above. Several examples of such functions exist in literature. The author of this note recently constructed functions of the latter type starting with a convergent series Σa_n of positive terms satisfying certain prescribed conditions. [Sen Gupta and Ganguli, (1959)].

In this context, it is natural that we should be looking for examples of (1, 1) continuous mappings of a closed n-dimensional interval onto a closed n-dimensional interval which is not measurable *i.e.*, a (1, 1) continuous mapping sending one interval onto the other such that there is at least one subset of measure zero of the former going onto a subset of positive measure of the latter.

In this note we consider such a mapping of the unit square $\{x, y\}$ $(0 \leq x \leq 1, 0 \leq y \leq 1)$ onto the unit square $\{X, Y\}$, $(0 \leq X \leq 1, 0 \leq Y \leq 1)$, the method employed being applicable to mappings in n-dimensional spaces. Further, the method is such that given any prescribed point (x, y) of the unit square in the xy-plane, we can determine the point (X, Y) in the XY-plane to which it is sent by the mapping.

Let us take a function $X = f(x)$ such that the function $f(x)$ is steadily increasing and continuous but not absolutely continuous in $0 \leq x \leq 1$ with $f(0) = 0$ and $f(1) = 1$.

We define such a function as follows:

Let $\Sigma a_n = 1$, $a_n > 0$ such that

$$0 < a_n - r_n < b_n/2^{n-1}$$

where $$r_n = a_{n+1} + a_{n+2} + \cdots \cdots$$

and $$\Sigma b_n = c < 1, \quad (b_n > 0)$$

Let S be the Cantor set on $0 \leq x \leq 1$ so that points of S can always be expressed uniquely in the form

$$\cdot d_1 d_2 d_3 \ldots \ldots \quad \text{(scale of 3)}$$

where each $d_n = 0$ or 2.

If $x \in S$ and if $x = \cdot d_1 d_2 d_3 \ldots \ldots$ (scale of 3)

where d_n is 0 or 2, we define $f(x)$ by

$$f(x) = \Sigma a'_n \text{ where } a'_n = a_n \text{ or } 0 \text{ acc. as } d_n = 2 \text{ or } 0.$$

The function $f(x)$ so defined on S is steadily increasing and continuous on S with

$$f(0) = 0, \; f(1) = 1.$$

If (ξ, η), $\xi < \eta$, be an interval constituting the complementary set CS, the above process gives $f(\xi) < f(\eta)$. In the complementary intervals we define $f(x)$ linearly. Clearly ξ and η have representations of the form

$$\xi = \cdot u_1 u_2 \ldots u_k \; 0 \dot{2}$$

$$\eta = \cdot u_1 u_2 \ldots u_k \; 2 \dot{0}$$

where $$u_r = 0 \text{ or } 2, \quad r = 1, 2, \ldots, k.$$

Obviously, $$f(\xi) = \sum_1^k a'_r + r_{k+1}$$

and $$f(\eta) = \sum_1^k a'_r + a_{k+1}$$

If x lies between ξ and η so that x is of the form

$$x = \xi + \theta(\eta - \xi), \; 0 < \theta < 1$$

we define $f(x)$ by

$$f(x) = f(\xi) + \theta[f(\eta) - f(\xi)]$$

$$= \sum_1^k a'_r + r_{k+1} + \theta[a_{k+1} - r_{k+1}]$$

The function $f(x)$ thus defined on $0 \leq x \leq 1$ has been proved (Sen Gupta and Ganguli, 1959) to be steadily increasing and continuous but not absolutely continuous in $0 \leq x \leq 1$ and

$$mf(s) = \lim_{n \to \infty} 2^n r_n > 0$$

Next we construct a similar function $g(x)$ on $0 \leq x \leq 1$ as follows.

Let $\Sigma A_n = 1$, $A_n > 0$ such that

$$0 < A_n - R_n < B_n / 2^{n \; 1}$$

where $$R_n = A_{n+1} + A_{n+2} + \ldots\ldots$$

and $$\Sigma B_n = C < 1, \; B_n > 0$$

If $x \in S$, and $x = \cdot d_1 d_2 d_3 \ldots\ldots$(scale 3)

where $d_n = 2$ or 0, we take, as before,

$$g(x) = \Sigma A'_n \text{ where } A'_n = A_n \text{ or } 0 \text{ acc. as } d_n = 2 \text{ or } 0$$

In a complementary interval $\xi < x < \eta$ where ξ and η are necessarily of the form

$$\xi = \cdot u_1 \, u_2 \ldots u_k \; 0 \, \dot{2}$$

$$\eta = u_1 \, u_2 \ldots u_k \; 2 \, \dot{0}$$

with

$$u_r = 0 \text{ or } 2, \; r = 1, 2, \ldots, k$$

we define $g(x)$, for $x = \xi + \theta'(\eta - \xi)$, $0 < \theta' < 1$, by

$$g(x) = \sum_1^k A'_r + R_{k+1} + \theta'(A_{k+1} - R_{k+1})$$

The function $g(x)$, as defined above on $0 \leq x \leq 1$, is steadily increasing and continuous but not absolutely continuous in $0 \leq x \leq 1$ and $mg(s) = \lim_{n \to \infty} 2^n R_n > 0$.

We now take the unit square S_{xy}, $0 \leq x \leq 1, 0 \leq y \leq 1$, in the xy-plane and consider its map on the XY-plane by the transformation

$$X = f(x) \qquad 0 \leq x \leq 1$$

$$Y = g(y) \qquad 0 \leq y \leq 1$$

By the above transformation, each point of the unit square S_{XY} goes to a point (X, Y) of the unit square S_{XY} $(0 \leq X \leq 1,\ 0 \leq Y \leq 1)$ of the XY-plane. We shall show that to each point (X, Y) of the unit square S_{XY} there corresponds one and only one point (x, y) of the unit square in the xy-plane so that the above mapping is onto and one-one.

Let (X, Y) be a given point of S_{XY}. Since $f(x)$ is steadily increasing and continuous in $0 \leq x \leq 1$ with $f(0) = 0$ $f(1) = 1$, there is one and only one point $\xi (0 \leq \xi \leq 1)$ such that $X = f(\xi)$ or $\xi = f^{-1}(X)$. Similarly, there is one and only one point $\eta (0 \leq \eta \leq 1)$ such that $Y = g(\eta)$ or $\eta = g^{-1}(Y)$. Thus there exists one and only one point (ξ, η) in S_{xy} which goes over to the given point (X, Y) of the unit square S_{XY} under the given transformation. Hence the transformation

$$X = f(x)$$

$$Y = g(y)$$

gives rise to a one-one mapping of S_{xy} onto S_{XY}.

Again, since $X = f(x)$ and $Y = g(y)$ are continuous in $0 \leq x \leq 1$ and $0 \leq y \leq 1$ respectively, the above transformation throws an open set of S_{xy} into an open set of S_{XY}; and since the inverse functions $x = f^{-1}(X)$ and $y = g^{-1}(Y)$ are also continuous, an open set of S_{XY} goes over to an open set of S_{xy}.

We now consider the product set $S \times S$ which consists of aggregates of pairs

$$(\cdot d_1 d_2 d_3 \ldots, \; \cdot e_1 e_2 e_3 \ldots)$$

wheres each d_n and e_n take the values 0 and 2 at random. It forms a non-dense set of Lebesgue plane measure zero. It will now be shown that under the transformations

$$X = f(x)$$

$$Y = g(y)$$

the set $S \times S$ is mapped into a set of positeve measure.

It is easy to verify that $f(1/3^k) = r_k$ and $g(1/3^k) = R_k$. The square $\{(x, y)\}$, $0 \leq x \leq \frac{1}{3}$, $0 \leq y \leq \frac{1}{3}$, therefore, goes over to the rectangle $\{(X, Y)\}$, $0 \leq X \leq r_1$, $0 \leq Y \leq R_1$. Hence the area of the map of the square $(0 \leq x \leq \frac{1}{3}, 0 \leq y \leq \frac{1}{3})$ is $r_1 R_1$. Similarly, the area of the map of the square $0 \leq x \leq 1/3^2$, $0 \leq y \leq 1/3^2$ is $r_2 R_2$ and so on. It follows that the measure of the image of the set $S \times S$ is

$$\lim_{n \to \infty} (4^n r_n R_n) = \lim (2^n r_n \cdot 2^n R_n) > 0$$

Thus the set $S \times S$, which is of measure zero, is mapped by $X = f(x)$, $Y = g(y)$ into a set of positive measure. It follows that the mapping induced by the transformation

$$X = f(x)$$
$$Y = g(y)$$

is non-measurable.

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ON THE ISOTONE MAPPING OF A LATTICE

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1. The object of this paper is to establish a connection between a lattice and an m-lattice by considering the isotone mappings of the lattice into itself. Such isotone mappings are endomappings of the lattice One can deal with these mappings just in the same manner as with the endomorphisms of an abelian group which form a ring. One can prove here that the isotone mappings of a lattice form a doubly distributive m-lattice. To get this result, it will be useful to start from the more elementary system, namely an upper semilattice.

A system P having a reflexive, antisymmetric and a transitive relation is called a poset. If every pair of elements of P has a least upper bound then P is called an upper semilattice. If one denote the least upper bound by the sum $a+b$ of two elements a, b, then an upper semilattice is a commutative additive semigroup in which every element is idempotent. For, as the sum is the least upper bound, $a \sqsupseteq b$ imply that $a \sqsupseteq a+b$ and $a+b \sqsupseteq a$. Hence $a \sqsupseteq b$ imply that $a=a+b$. The commutative law and the idempotency are evident. To prove the associative law, one notes that $(a+b)+c \sqsupseteq a, b, c$ and hence $(a+b)+c \sqsupseteq a+(b+c)$. Similarly one can prove that $a+(b+c) \sqsupseteq (a+b)+c$. Thus the associative law is true. Conversely in a commuative additive semigroup in which every element is idempotent, one defines an inclusion by the condition that $a \sqsupseteq b$ if and only if $a=a+b$. Then, as $a+a=a$, the inclusion is reflexive and as the commutative law holds, $a=a+b$, and $b=b+a$ imply that $a=b$. This shows that the inclusion is antisymmetric. Similarly the transitivity of the inclusion follows from the associative law. Lastly if $c \sqsupseteq a$, $c \sqsupseteq b$ then $a+c=c$ and $b+c=c$ and hence $(a+b)+c=a+(b+c)=a+c=c$. Thus $c \sqsupseteq a+b$; this proves that $a+b$ is the least upper bound of a, b. Hence the given semigroup is an upper semilattice.

A mapping of an upper semilattice U into itself is called an endomapping. A mapping which keeps the inclusion invariant is called isotone and those which reverses it, is called antitone. Let f be an isotone mapping of an upper semilattice U on to another upper semilattice $f(U)$; then as $a+b \sqsupseteq a, b$ in U, $f(a+b) \sqsupseteq f(a), f(b)$ in $f(U)$. Hence, as $f(U)$ is an upper semilattice, $f(a+b) \sqsupseteq f(a)+f(b)$. A mapping of an upper semilattice which satisfies this condition is called an upper homomorphism. Conversely if f is an upper homomorphism of U onto $f(U)$, then f is isotone. For, if $a \sqsubseteq b$, $a+b=b$ and $f(a) \sqsubseteq f(a)+f(b) \sqsubseteq f(a+b)=f(b)$. An endomapping which is an upper homomorphism is called an upper endomorphism.

Notions dual to that of an upper semilattice can be defined. If every pair of elements of a poset has a greatest lower bound, then the poset is called a lower semilattice. If the greatest lower bound of two elements a, b is denoted by ab, then a lower semilattice is a commutative multiplicative semigroup in which every element is idempotent. An isotone mapping of a lower semilattice is a lower homomorphism: $f(ab) \subseteq f(a)f(b)$.

A poset which is both a lower and an upper semilattice is called a lattice. An isotone mapping of a lattice satisfies the inclusions: $f(a+b) \supseteq f(a)+f(b)$; $f(ab) \subseteq f(a)f(b)$. Now a homomorphism of a semilattice is isotone. Hence the equality in one of the above inclusion does not imply the equality in the other. Thus there are three special cases: homomorphism as a lower semilattice, homomorphism as an upper semilattice and homomorphism as a lattice. The first type will be called l-homomorphism, the second type as u-homomorphism and last type as homomorphism of the lattice.

2. Let us now consider the endomappings of an upper semilattice U whose elements will be denoted by the Greek letters α, β, γ etc. The endomappings of U will be denoted by the Roman letters a, b etc. If a is an endomapping, then $a(\alpha) = \beta \varepsilon U$. For simplicity, $a(\alpha)$ will be written as $a\alpha$. The set of all endomappings of U will be denoted by Ω. Among the elements of Ω, define a sum $a+b$ by the rule $(a+b)\alpha = a\alpha + b\alpha$.

This definition is evidently admissible as U is an upper semilattice. Then one can prove the

Theorem. *Ω is an upper semilattice.*

Proof. For every $\sigma \varepsilon U$ and given $a \varepsilon \Omega$, $(a+a)\alpha = a\sigma + a\alpha = a\alpha$. Hence $a+a = a$.

Also $(a+b)\alpha = a\alpha + b\alpha = b\alpha + a\alpha = (b+a)\alpha$ for all α. Hence $a+b = b+a$.

Lastly $[(a+b)+c]\alpha = (a+b)\alpha + c\alpha = a\alpha + b\alpha + c\alpha = a\alpha + (b\alpha + c\alpha) = [a+(b+c)]\alpha$.

Hence $(a+b)+c = a+(b+c)$. This proves the theorem.

In Ω, the inclusion is given by the rule:

$a+b = a$ if and only if $(a+b)\alpha = a\sigma$, *i e.*, $a\alpha + b\alpha = a$ as σ.

Thus $a \supseteq b$ if and only if $a\alpha \supseteq b\alpha$ for every σ.

This inclusion is called the strength of the mapping and if $a \supseteq b$, a is called stronger than b and b is called weaker than a.

Let us denote by a_α the mapping $\xi \to \alpha$ where ξ is the general element of U and α is a given element of U. Then a_1 is the topmost element of Ω *i e.*, $a \subseteq a_1$. For, $a\xi \subseteq a_1\xi = 1$. Ω also contains the identity mapping $i : \xi \to \xi$ Then $i\xi = \xi$.

As $a\xi$ and ξ are both elements of U, they are either (*i*) incomparable, (*ii*) $a\xi \supseteq \xi$ or (*iii*) $a\xi \subseteq \xi$. Hence a and i are either (*i*) incomparable, (*ii*) $a \supseteq i$ (*iii*) $a \subseteq i$. In (*ii*) a is called inclusive and in (*iii*) a is called exclusive Denote the set of inclusive mappings by Ω_i and that of exclusive mappings by Ω_e. Then Ω_i and Ω_e are upper semilattices. For, if a, $b \varepsilon \Omega_e$, $a \subseteq i$, $b \subseteq i$.

Hence $a+b \subseteq i$. Also if a, $b \varepsilon \Omega_i$, $a_1 \supseteq a$, $b \supseteq i$ and hence $a_1 \supseteq a+b \supseteq i$.

3. One can now define the product ab of two mappings a and b by the rule $(ab)\xi = a(b\xi)$. This product is associative. For,

$$[(ab)c]\xi = (ab)(c\xi) = a[b(c\xi)] = a[(bc)\xi] = [a(bc)]\xi$$

Hence $$(ab)c = a(bc).$$

Also $[(a+b)c]\xi = (a+b)(c\xi) = (a+b)\xi' = a\xi'+b\xi' = a(c\xi)+b(c\xi) = (ac)\xi+(bc)\xi = (ac+bc)\xi$ where $\xi' = c\xi$. Thus the product is right distributive. But the left distributive law is not true in general. Indeed one can prove the proposition:

If c is isotone, then $c(a+b) \sqsupseteq ca+cb$ for every pair of endomappings and conversely.

For, if c is isotone, then $[c(a+b)]\xi = c[(a+b)\xi] = c(a\xi+b\xi) \sqsupseteq c(a\xi)+c(b\xi) = (ca)\xi + (cb)\xi = (ca+cb)\xi$.

Hence $$c(a+b) \sqsupseteq ca+cb.$$

This inclusion will be called as the left distributive inclusion.

Conversely let left distributive inclusion hold for every pair of endomappings a, b and a given mapping c. Then, operating on the general element ξ of U,

$$[c(a+b]\xi \sqsupseteq (ca+cb)\xi \text{ i.e. } c(a\xi+b\xi) \sqsupseteq c(a\xi)+c(b\xi)$$

Putting $$a\xi = \xi_1,\ b\xi = \xi_2,\ c(\xi_1+\xi_2) \sqsupseteq c\xi_1 + c\xi_2.$$

The mapping c is thus an upper homomorphism and hence c is isotone.

If follows, as a corrolary that $c(a+b) = ca+cb$ if and only if c is a homomorphism.

The set of idempotent endomappings of U will be denoted by Ω_d. The identical mapping i is idempotent, for $i^2 = i$. Also a_1 is idempotent. Indeed $a_1{}^2\xi = a_1\xi = 1$ and $a_1\xi = 1$. Thus both a_1 and $a_1{}^2$ map ξ into 1. It is evident that $ia = a = ai$. But $a_1a = a_1$ and $aa_1 \neq a_1$. Define in an inclusion $a \sqsubseteq b$ if and only if $ab = a$. Then as $a_1a = a_1$, $a_1 \sqsubseteq a$ and as $ai = a$, $a \sqsubseteq i$. Thus $a_1 \sqsubseteq a \sqsubseteq i$. The inclusion in Ω_d is reflexive as $a^2 = a$. Also if $a \sqsubseteq b$, $b \sqsubseteq c$, then $ab = a$ and $bc = b$, so that $ac = (ab)c = a(bc) = ab = a$. Hence $a \sqsubseteq c$ and the inclusion is transitive. Ω_d becomes thus quasi-ordered by the inclusion. If the product is commutative, then the above inclusion is antisymmetric and Ω_d is a poset and is a lower semilattice. The absorptive laws need not hold here and consequently Ω_d is not necessarily a lattice.

A system Ω is called an upper m-semilattice if (i) it is an upper semilattice with a sum $a+b$ and (ii) is a right distributive semigroup for a product ab. Using the definition, one can state the above result as the.

Theorem. *The endomappings of an upper semilattice form an upper m-semilattice. The isotone endomappings satisfy the left distributive inclusion. The endomorphisms are left distributive.*

One can state the dual results by taking a lower semilattice L in which the product will be denoted by $a \times b$. In this case, a system Ω will be called a lower m-semilattice if (i) it is a lower semilattice with product $a \times b$ and (ii) is a right distributive semigroup for a product ab. The right distributive law is here $(a \times b)c = ac \times bc$, The mapping $a_0:\xi\to 0$ is the lowest element in Ω. One has therefore the theorem:

The endomappings of a lower semilattice form a lower m-semilattice. The isotone mappings satisfy the left distributive inclusion. Then endomorphisms are left distributive.

One can generalise the notion of the upper m-semilattice into the notion of an m-lattice. A system is called an m-lattice if it is (*i*) an upper m-semilattice with a sum $a+b$ and a product ab, and (*ii*) a lattice with the same sum $a+b$ and a product $a\times b$. This lattice product will be called the primary product and the product ab will be called the secondary product. This definition of an m lattice is somewhat restricted than the one given in Birkhoff (p.100) as the associative law is assumed for the secondary product ab.

As the m-lattice is an upper m-semilattice, the right distributive law hold: $(a+b)c = ac+bc$. Now if $a\subseteq b$, then $a+b = b$ and by the right distributive law, $ac+bc = bc$ Hence $ac\subseteq bc$ which proves that the right multiplication is isotone. As $a\times b\subseteq a, b$, it follows that $(a\times b)c\subseteq ac\times bc$. Thus the right distributive inclusion holds for the lattice product. If $(a\times b)c = ac\times bc$, the m-lattice is called doubly right distributive. One can now prove the

Theorem. *The endomappings of a lattice L form a doubly right distributive m-lattice Ω.*

Proof. As L is an upper as well as a lower semilattice, the set Ω of endomappings of L is an upper as well as a lower m-semilattice. The absorptive laws hold. For, $[(a+b)\times a]\xi = (a\xi+b\xi)\times a\xi = a\xi$ by the absorptive law in L. Thus $(a+b)\times a = a$. Similarly $a+b\times a = a$ is proved. Hence Ω is a lattice. As Ω is both an upper and a lower m-semilattice, the two right distributive laws hold. Thus Ω is a doubly right distributive m-lattice.

In the m-lattice of endomappings of a lattice L, the left distributive inclusion $c(a+b)\supseteq ca+cb$ imply that c is isotone which again imply left distributive inclusion $c(a\times b)\subseteq ca\times cb$. Thus $c(a+b)\supseteq ca+cb$ imply $c(a\times b)\subseteq ca\times cb$ and conversely. Here the equality holds when c is any one of the three types of homomorphism. Thus one gets four types of the m-lattices of endomappings of a lattice L: one in which (*i*) $c(a+b)=ca+cb$, (*ii*) $c(a\times b) = ca\times cb$, (*iii*) $c(a+b) = ca+cb$, $c(a\times b) = ca\times cb$ or (*iv*) $c(a+b)\supset ca+cb$, $c(a\times b)\subset ca\times cb$. c is a u-homomorphism of the lattice in (*i*) and a l-homomorphisms in (*ii*). In (*iii*) c is a lattice homomorphism and in (*iv*) c is strictly isotone.

In the case (*iii*), both left and right sided distributive laws are true. All mappings are homomorphisms of the lattice. The m-lattice is called doubly distributive in this case.

One can prove the interesting result:

$(a\times b)(a+b)\subseteq ab+ba$ in the m-lattice of u-homomorphisms. For, as the right multiplication is isotone and $a\times b\subseteq a, b$

$$(a\times b)a\subseteq ba,\quad (a\times b)b\subseteq ab.$$

Adding and using the left distributive law, $(a\times b)(a+b)\subseteq ab+ba$. Similarly. $(a\times b)(a+b)\supseteq ab\times ba$ in the m-lattice of l-homomorphisms. An m-lattice is called commutative if the secondary product is commutative. A commutative doubly right distributive m-lattice is doubly distributive and hence is an m-lattice of homomorphisms. Then in the m-lattice of homomorphisms both the above inclusions hold and hence $(a+b)(a \times b) = ab$ in it.

If the secondary product satisfies the cancellation law:

$$ax = bx \text{ or } xa = xb \text{ implies } a = b,$$

then the m-lattice is called cancellable. A commutative cancellable m-lattice will be called an integral m-lattice. It is well known that in an integral m-lattice, one of the distributive law can be replaced by the law: $(a+b)(a \times b) = ab$. Further its lattice is distributive (Choudhury, 1957).

4. The upper m-semilattice Ω of the endomappings of an upper semilattice U contains a sub-semilattice isomorphic to U. Thus U can be embeded in Ω. For the purpose of proof, denote the mapping $\xi \leftrightarrow \alpha$ by a_α where ξ is the general element of U and α is a fixed element of U. Then the mapping $a_\alpha \longleftrightarrow \alpha$ is biunique. Also as $a_\alpha \xi = \alpha$, $a_\beta \xi = \beta (a_\alpha + a_\beta)\xi = a_\alpha \xi + a_\beta \xi = \alpha + \beta$. If $\alpha \supseteq \beta$, $\alpha + \beta = \alpha$ and hence $(a_\alpha + a_\beta)\xi = \alpha = a_\alpha \xi$. Thus $a_\alpha + a_\beta = a_\alpha$ *i.e.*, $a_\alpha \supseteq a_\beta$. On the other hand, if $a_\alpha \supseteq a_\beta$, $a_\alpha + a_\beta = a_\alpha$ and hence $(a_\alpha + a_\beta)\xi = a_\alpha \xi$ *i.e.* $\alpha + \beta = \alpha$ so that $\alpha \supseteq \beta$. Thus the mapping $a_\alpha \longleftrightarrow \alpha$ is bisotone. As this mapping is biunique, it is an isomorphism. Consequently, the mappings a_α form an upper semilattice isomorphic to U. If one identifies a_α with α, then U can be considered as a subset of Ω. Thus U can be embedded in Ω.

One should note here that when U is embedded in Ω, $a(\alpha)$ can be expressed as $a\alpha$ and this means that the operational notation has been expressed as the secondary multiplication. Also as the mapping a_α is not isotone, U remain outside the Ω_i of isotone mappings of Ω, when embedded in Ω.

Let us now construct an example of an m-lattice. The boolean algebra 2 is formed by two elements 0 and 1. The endomappings of 2 are expressible as matrices

$$1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here the matrix $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$ is the mapping by which $0 \to \alpha$, $1 \to \beta$.

Then $i\alpha = \alpha$, $1\alpha = 1$, $0\alpha = 0$, $a\alpha = \beta \neq \alpha$. i is the identity mapping, 1 is the mapping a_1 and 0 is the mapping a_0 of the above notation. Here $1 \supseteq i \supseteq 0$ and $1 \supseteq a \supseteq 0$. The mappings i and a are incomparable. Indeed, i is isotone and a is antitone. Thus the lattice of m-lattice of endomappings is the rectangular lattice. The table for the secondary product is

	0	i	a	1
0	0	0	0	0
i	0	i	a	1
a	1	a	i	0
1	1	1	1	1

One notes here that $a(1+i) = a0 = 0$, $a1+ai = 0+a = a$. As $a \supseteq 0$, $a(1+i) \subseteq a1+ai$. This is true as a is antitone. More generally if a is antitone, $a(\alpha+\beta) \subseteq a\alpha+a\beta$. But the converse is not true always · Indeed it may happen that α, β are incomparable, but $a\alpha$, $a\beta$ are comparable. Thus one gets as before that for antitone mappings, $a(b+c) \subseteq ab+ac$ though the converse is not true.

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ON THE OSCILLATION OF A CIRCULAR CYLINDER IN A VISCOUS LIQUID CONTAINED IN A CO-AXIAL CIRCULAR CYLINDER

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The flow of a viscous liquid in an annulus with porous walls has been investigated by Berman (1958). In this paper the flow of a viscous liquid between two co-axial porous circular cylinders when the inner cylinder is oscillating is discussed. The solution of the problem has been obtained in terms of Hankel functions and from it the results in the two extreme cases of very small and very large frequencies are deduced. As a special case, putting the radial component of the velocity equal to zero, the solution is obtained for the oscillatory motion of a piston in a viscous liquid contained in a circular cylinder.

1. We shall consider the oscillatory motion of a viscous liquid confined between two concentric circular cylinders of radii a and $b(b > a)$. Let us assume that the rate of liquid withdrawn at one wall of the annulus is always equal to the rate of injection of the liquid at the other wall and that these rates are independent of the axial position in the tube. If we select a cylindrical co-ordinates x, r, θ, x-axis coinciding with the common axis of the cylinders and if u, v, w be the components of velocity in the directions of x, r, θ increasing respectively then evidently $w = 0$. All the physical quantities are independents of x and θ. The equations of motion and continuity are the followiug:—

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r} = -\frac{1}{\varrho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right) \tag{2}$$

$$\frac{\partial v}{\partial r}+\frac{v}{r} = 0 \tag{3}$$

where ν is the kinematic viscosity and ϱ the density of the liquid.

The condition that the suction rate at one wall be equal to the injection rate at the other wall is satisfied if

$$b.v_b = a.v_a \tag{4}$$

where v_a and v_b are the radial velocities at the walls of the smaller and larger tubes respectively.

The contiuuity equation (3) and equation (4) then give

$$v.r = a.v_a = b.v_b$$

$$\therefore \quad v = \frac{av_a}{r} = \frac{b.v_b}{r}. \tag{5}$$

Hence from (2), assuming v_a = constant, we get

$$\frac{p-p_0}{\varrho} = \frac{v^2}{2}\left(1-\frac{a^2}{r^2}\right) \tag{6}$$

where $p = p_0$ when $r = a$.

We can now write the equation (1) as

$$\frac{\partial u}{\partial t} + \frac{a\,v_a}{r}.\frac{\partial u}{\partial r} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r}\right) \tag{7}$$

To obtain a solution of this equation, we put

$$u(r,\ t) = f(r).\ \exp\ (i\sigma t) \tag{8}$$

where the real part of the right hand side is taken.

The equation (7) reduces to

$$\frac{d^2 f}{dr^2} + \frac{1-R}{r}.\frac{df}{dr} + h^2.f = 0 \tag{9}$$

where $R = av_a/\nu$ is cross flow Reynolds number and $h^2 = -i\sigma/\nu$.

Substituting

$$f(r) = (r)^{\frac{1}{2}R}F(r) \tag{10}$$

in (9), we get

$$\frac{d^2F}{dr^2} + \frac{1}{r}\,\frac{dF}{dr} + \left[h^2 - \frac{(\frac{1}{2}R)^2}{r^2}\right]F = 0. \tag{11}$$

This equation is Bessel's equation of the $(R/2)$th order, the independent variable being hr.

An integral of (11) can be written as

$$F(r) = A.H^1_{R/2}(hr) + B.H^2_{R/2}(hr) \tag{12}$$

where $H^1_{R/2}(hr)$ and $H^2_{R/2}(hr)$ are Hankel functions of the first and second kinds respectively of the argument hr. Hence by (10) and (8)

$$u(r,\ t) = (r)^{R/2}.\ \mathbf{Re}[\{AH^1_{R/2}(hr) + B.H^2_{R/2}(hr)\}\ \exp.(i\sigma t)] \tag{13}$$

where **Re** denotes the real part. The boundary conditions are

$$\left.\begin{aligned} u &= U\,\mathbf{Re}\,[\exp\ (i\sigma t] && \text{when } r = a \\ &= 0 && \text{when } r = b \end{aligned}\right\} \tag{14}$$

where U is a real constant and **Re** denotes the real part. These conditions give

$$[A.H^1_{R/2}(ha) + B\ H^2_{R/2}(ha)]a^{R/2} = U$$

$$[A.H^1_{R/2}(hb) + B.H^2_{R/2}(hb)]b^{R/2} = 0.$$

Solving these equations, we get

$$A = -\frac{H^2_{R/2}(hb)\,Ua^{-R/2}}{H^1_{R/2}(hb).H^2_{R/2}(ha) - H^2_{R/2}(hb).H^1_{R/2}(ha)}$$

$$B = \frac{H^1_{R/2}(hb)\,Ua^{-R/2}}{H^1_{R/2}(hb).H^2_{R/2}(ha) - H^2_{R/2}(hb).H^1_{R/2}(ha)}.$$

Substituting in (13), we get

$$u(r,\ t) = U(r/a)^{\frac{1}{2}R}.\ \mathbf{Re}\left[\left\{\frac{H^1_{R/2}(hr).H^2_{R/2}(hb) - H^2_{R/2}(hr).H^1_{R/2}(hb)}{H^1_{R/2}(ha).H^2_{R/2}(hb) - H^2_{R/2}(ha).H^1_{R/2}(hb)}.\right\} \exp\,(i\sigma t).\right] \quad (15$$

2. **For small frequencies.** In terms of the Bessel functions $J_{R/2}(hr)$ and $J_{-R/2}(hr)$ we have (Sommerfeld, p. 91)

$$H^1_{R/2}(hr) = \frac{\exp\,(-i.R/2.\pi)J_{\frac{1}{2}R}(hr) - J_{-\frac{1}{2}R}(hr)}{-i\sin\frac{1}{2}R.\pi}$$

$$H^2_{R/2}(hr) = \frac{\exp\,(i.R/2.\pi)J_{\frac{1}{2}R}(hr) - J_{-\frac{1}{2}R}(hr)}{i\sin\frac{1}{2}R.\pi}$$

$$\therefore\ \frac{H^1_{R/2}(hr).H^2_{R/2}(hb) - H^2_{R/2}(hr).H^1_{R/2}(hb)}{H^1_{R/2}(ha).H^2_{R/2}(hb) - H^2_{R/2}(ha).H^1_{R/2}(hb)}$$

$$= \frac{J_{\frac{1}{2}R}(hb)J_{-\frac{1}{2}R}(hr) - J_{\frac{1}{2}R}(hr).J_{-\frac{1}{2}R}(hb)}{J_{\frac{1}{2}R}(hb).J_{-\frac{1}{2}R}(ha) - J_{\frac{1}{2}R}(ha).J_{-\frac{1}{2}R}(hb)}. \qquad (16)$$

For small frequencies, the absolute value of hr is small. We have (Sommerfeld, p.92).

$$J_{\frac{1}{2}R}(z) = (\tfrac{1}{2}z)^{\frac{1}{2}R}\left[\frac{1}{\Gamma(R/2+1)} - \frac{z^2}{4\Gamma(R/2+2)} + \cdots\right] = (\tfrac{1}{2}z)^{\frac{1}{2}R}(a_0 - a_1 z^2 + \ldots)$$

$$J_{-\frac{1}{2}R}(z) = (\tfrac{1}{2}z)^{-\frac{1}{2}R}\left[\frac{1}{\Gamma(-\frac{1}{2}R+1)} - \frac{z^2}{4\Gamma(-\frac{1}{2}R+2)} + \cdots\right] = (\tfrac{1}{2}z)^{-\frac{1}{2}R}(b_0 - b_1 z^2 + \ldots)$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\Gamma(R/2+1)}, & a_1 &= \frac{1}{4.\Gamma(\frac{1}{2}R+2)} \\ b_0 &= \frac{1}{\Gamma(-R/2+1)}, & b_1 &= \frac{1}{4.\Gamma(-R/2+2)} \end{aligned}\right\} \qquad (17)$$

Hence neglecting terms containing powers of h higher than two, we have

$$
\begin{aligned}
J_{\frac{1}{2}R}(hq).J_{-\frac{1}{2}R}(hp) - J_{\frac{1}{2}R}(hp).J_{-\frac{1}{2}R}(hq)
&= \left(\frac{q}{p}\right)^{\frac{1}{2}R}.(a_0b_0 - a_0b_1h^2p^2 - a_1b_0h^2q^2) \\
&\quad - \left(\frac{p}{q}\right)^{\frac{1}{2}R}.(a_0b_0 - a_0b_1h^2q^2 - a_1b_0h^2p^2) \\
&= a_0b_0\left[\left(\frac{q}{p}\right)^{\frac{1}{2}R} - \left(\frac{p}{q}\right)^{\frac{1}{2}R}\right] - h^2\left[(a_0b_1p^2 + a_1b_0q^2)\left(\frac{q}{p}\right)^{\frac{1}{2}R}\right. \\
&\quad \left. - (a_0b_1q^2 + a_1b_0p^2)\left(\frac{p}{q}\right)^{\frac{1}{2}R}\right].
\end{aligned}
$$

Therefore, from (16), we get

$$
\begin{aligned}
&\frac{H^1_{R/2}(hr)H^2_{R/2}(hb) - H^2_{R/2}(hr)H^1_{R/2}(hb)}{H^1_{R/2}(ha)H^2_{R/2}(hb) - H^2_{R/2}(ha)H^1_{R/2}(hb)} \\
&= \frac{a_0b_0[(b/r)^{\frac{1}{2}R} - (r/b)^{\frac{1}{2}R}] - h^2[(a_0b_1r^2 + a_1b_0b^2)(b/r)^{\frac{1}{2}R} - (a_0b_1b^2 + a_1b_0r^2)(r/b)^{\frac{1}{2}R}]}{a_0b_0[(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}] - h^2[(a_0b_1a^2 + a_1b_0b^2)(b/a)^{\frac{1}{2}R} - (a_0b_1b^2 + a_1b_0a^2)(a/b)^{\frac{1}{2}R}]} \\
&= \frac{(b/r)^{\frac{1}{2}R} - (r/b)^{\frac{1}{2}R}}{(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}} - \frac{h^2}{a_0{}^2b_0{}^2[(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}]^2}\,[\{(a_0b_1r^2 + a_1b_0b^2)(b/r)^{\frac{1}{2}R} \\
&\quad - (a_0b_1b^2 + a_1b_0r^2)(r/b)^{\frac{1}{2}R}\}a_0b_0\{(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}\} - \{(a_0b_1a^2 + a_1b_0b^2)(b/a)^{\frac{1}{2}R} \\
&\quad - (a_0b_1b^2 + a_1b_0a^2)(a/b)^{\frac{1}{2}R}\}a_0b_0\{(b/r)^{\frac{1}{2}R} - (r/b)^{\frac{1}{2}R}\}]. \qquad (18)
\end{aligned}
$$

Since

$$h^2 = -\frac{i\sigma}{\nu}$$

we have from (15)

$$
\begin{aligned}
u(r,\ t) &= U(r/a)^{\frac{1}{2}R}\left[\frac{(b/r)^{\frac{1}{2}R} - (r/b)^{\frac{1}{2}R}}{(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}}\cos\sigma t - \frac{(\sigma/\nu)\sin\sigma t}{a_0{}^2b_0{}^2\{(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}\}^2}\right. \\
&\quad \times (\{(a_0b_1r^2 + a_1b_0b^2)(b/r)^{\frac{1}{2}R} - (a_0b_1b^2 + a_1b_0r^2)(r/b)^{\frac{1}{2}R}\}a_0b_0\{(b/a)^{\frac{1}{2}R} - (a/b)^{\frac{1}{2}R}\} \\
&\quad \left. - \{(a_0b_1a^2 + a_1b_0b^2)(b/a)^{\frac{1}{2}R} - (a_0b_1b^2 + a_1b_0a^2)(a/b)^{\frac{1}{2}R}\}a_0b_0\{(b/r)^{\frac{1}{2}R} - (r/b)^{\frac{1}{2}R}\})\right]. \qquad (19)
\end{aligned}
$$

3. **For large frequencies.** For very large frequencies, $|h|$ is very large and we have the asymptotic formulas (Sommerfeld, p. 160)

$$H^1_{R/2}(hp) = \left(\frac{2}{\pi hp}\right)^{\frac{1}{2}}.\ \exp\ [i\{hp - (\tfrac{1}{2}R + \tfrac{1}{2})\tfrac{1}{2}\pi\}]$$

$$H^2_{R/2}(hp) = \left(\frac{2}{\pi hp}\right)^{\frac{1}{2}}.\ \exp\ [-i\{hp - (\tfrac{1}{2}R + \tfrac{1}{2})\tfrac{1}{2}\pi\}]$$

we have

$$h = \left(-\frac{i\sigma}{\nu}\right)^{\frac{1}{2}} = \beta(1-i)$$

where

$$\beta = \left(\frac{\sigma}{2\nu}\right)^{\frac{1}{2}}$$

Hence if $q > p$, we have approximately,

$$H^1{}_{R/2}(hp).\quad H^2{}_{R/2}(hq) - H^2{}_{R/2}(hp).\quad H^1{}_{R/2}(hq)$$

$$= -\frac{2}{\pi h\,(pq)^{\frac{1}{2}}}.\quad \exp\,\{ih(q-p)\}$$

Substituting in (15), we get

$$u(r,t) = U.\left(\frac{r}{a}\right)^{R/2}.\quad \mathbf{Re}\left[\left(\frac{a}{r}\right)^{\frac{1}{2}}.\quad \exp\, i\{h(a-r)+\sigma t\}\right]$$

$$= U.\left(\frac{r}{a}\right)^{(R-1)/2}.\quad \{\exp\; -\beta(r-a)\}.\quad \cos\,\{\beta(r-a)-\sigma t\} \qquad (20)$$

This result shows that the influence of viscosity extends only a short distance from the wall of the oscillating cylinder.

4. If we put $v_a = v_l = 0$ then $R = 0$. In this case the two cylinders are rigid cylinders and the motion is due to the oscillation of a piston in a viscous liquid contained in a circular cylinder.

Putting $R = 0$, in (15), we have

$$u(r,t) = U.\quad \mathbf{Re}\left[\left\{\frac{H'_0(hr).\; H_0{}^2(hb) - H_0{}^2(hr).\; H_0{}'(hb)}{H_0{}'(ha).\; H_0{}^2(hb) - H_0{}^2(ha).\; H_0{}'(hb)}\right\}\exp\,(i\sigma t)\right] \qquad (21)$$

For small frequencis, u can be obtained from (19). As in the limit $R \to 0$, it becomes %, the limiting value is determined according to De L' Hospital's rule.

We have (Sommerfeld, p. 94)

$$\Gamma'(1) = -\gamma,\quad \Gamma'(2) = -\gamma+1$$

where γ is Euler's constant.

∴ when $R \to 0$, we have from (17),

$$a_0 \to 1,\quad b_0 \to 1,\quad a_1 \to \frac{1}{4},\quad b_1 \to \frac{1}{4}$$

and

$$a'_0 \to \frac{\gamma}{2},\quad b'_0 \to -\frac{\gamma}{2},\quad a_1{}' \to \frac{\gamma-1}{8},\quad b_1{}' \to -\frac{\gamma-1}{8}$$

where dashes denote differentiation with respect to R.

Hence from (19), proceeding to the limit $R \to 0$, we obtain

$$u(r,t) = U.\ \frac{\log \frac{b}{r}}{\log \frac{b}{a}}.\ \cos\sigma t + \frac{U\sigma.\ \sin\sigma t}{4\nu\left(\log\frac{b}{a}\right)^2}\left[(a^2-r^2)\log\frac{b}{a}.\ \log\frac{b}{r}\right.$$

$$\left.+(b^2-r^2)\log\frac{b}{a}-(b^2-a^2)\log\frac{b}{r}\right] \qquad (22)$$

The retarding force per unit area on the oscillating cylinder is

$$-\mu\left(\frac{\partial u}{\partial r}\right)_{r=a} = \frac{\mu U}{a}\cdot\frac{\cos\sigma t}{\log\frac{b}{a}}+\frac{\varrho U\sigma}{4a\left(\log\frac{b}{a}\right)^2}\left[2a^2\left(\log\frac{b}{a}\right)^2+2a^2\log\frac{b}{a}\right.$$

$$\left.-(b^2-a^2)\right]\sin\sigma t. \qquad (23)$$

For large frequencies, putting $R = 0$ in (20), we get

$$u(r,t) = U\left(\frac{a}{r}\right)^{\frac{1}{2}} \exp\{-\beta(r-a)\}.\ \cos\{\beta(r-a)-\sigma t\} \qquad (24)$$

The retarding force per unit area on the oscillating cylinder is

$$-\mu\left(\frac{\partial u}{\partial r}\right)_{r=a} = \frac{\mu U}{a}\cos\sigma t+\mu U\beta(\cos\sigma t-\sin\sigma t) \qquad (25)$$

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NOTE ON THE TORSIONAL OSCILLATIONS OF DISCS OF SOME SIMPLE EXPONENTIAL PROFILES

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Introduction. In this note some *possible* symmetrical modes of torsional oscillations of discs with simple exponential profiles have been considered. Assuming that the discs have no rotation, the case of discs with fixed rims has been discussed

Method of solution. The general equation of all rotationally symmetrical oscillations (Biezeno and Grammel, 1955) is

$$\theta' + \left(\frac{y'}{y} + \frac{3}{r}\right)\theta' - \frac{\gamma}{gG}\ddot{\theta} = 0 \tag{i}$$

where r is the distance of a narrow circular strip of width dr of the disc from the centre, y is the thickness of that strip and θ is one of the two components into which the angle at the centre of the disc corresponding to the tangential displacement is split up. Also γ stands for the density, G for the modulus of rigidity of the disc and g for the acceleration due to gravity.

Dashes represent derivatives with respect to r and dots represent derivatives with respect to time.

We put

$$\theta = \psi \sin 2\pi\alpha t \tag{ii}$$

where α is the unknown frequency to be found and ψ is a function of r.

Equation (i) then reduces to

$$\psi'' + \left(\frac{y'}{y} + \frac{3}{r}\right)\psi' + \lambda^2\psi = 0 \tag{iii}$$

where

$$\lambda = 2\pi\alpha\sqrt{\frac{\gamma}{gG}}. \tag{iv}$$

Biezeno and Grammel (1955) considered oscillations of discs with some particular profiles like $y = c/r^n$ for simplicity of solution. Here also for the same reason some possible symmetrical modes of torsional oscillations of discs with exponential profiles given by

$$y = Ae^{-kr}$$

(k being a real number) have been considered.

For a disc with exponential profile

$$y = Ae^{-kr}$$

where k is any real number, equation (iii) becomes

$$\frac{d^2\psi}{dr^2}+\left(-k+\frac{3}{r}\right)\frac{d\psi}{dr}+\lambda^2\psi = 0. \tag{v}$$

Solution of (v) is given by

$$\psi = e^{-\frac{1}{2}\{-k+\beta\}r}\,\overline{M}\left[\frac{3}{2}\left(1-\frac{k}{\beta}\right),\ 3,\ \beta r\right] \tag{vi}$$

where
$$\beta = \sqrt{k^2-4\lambda^2}. \tag{vii}$$

We may take the solution as

$$\psi = e^{-\frac{1}{2}\{-k+\beta\}r}\,M\left[\frac{3}{2}\left(1-\frac{k}{\beta}\right),\ 3,\ \beta r\right], \tag{viii}$$

M denoting confluent hypergeometric function. For a fixed rim $r = a$, $\theta = 0$ at $r = a$, *i.e.*, $\psi = 0$ at $r = a$.

Hence
$$M\left[\frac{3}{2}\left(1-\frac{k}{\beta}\right),\ 3,\ \beta a\right] = 0 \tag{ix}$$

is the frequency equation.

We consider the following simple profiles.

(1) We put

$$k = 5/a \quad\text{and}\quad \beta = 3/a.$$

The lefthand side of (ix) becomes $M(-1, 3, 3)$ which is zero. (Math. Tables Commission, 1927). From (vii) we get

$$\lambda = 2/a.$$

Hence

$$\alpha = \frac{1}{\pi a}\sqrt{\frac{gG}{\gamma}} \tag{x}$$

giving a *possible* frequency when $k = 5/a$.

(2) We put $k = 14/(3a)$ and $\beta = 2/a$. Then the lefthand side of (ix) becomes $M(-2, 3, 2)$ which is zero (Math. Tables Commission, 1927).

From (vii) we get

$$\lambda = \frac{2\sqrt{10}}{3a}$$

Hence
$$\alpha = \frac{\sqrt{10}}{3\pi a}\sqrt{\frac{gG}{\gamma}} \tag{xi}$$

which gives a *possible* frequency when $k = 14/(3a)$.

(3) We put $k = 14/a$ and $\beta = 6/a$. The lefthand side of (ix) becomes $M(-2, 3, 6)$ which is zero. (Math. Tables Commission, 1927).

From (vii) we get

$$\lambda = \frac{2\sqrt{10}}{a}$$

Hence
$$\alpha = \frac{\sqrt{10}}{\pi a}\sqrt{\frac{gG}{\gamma}} \tag{xii}$$

giving a *possible* frequency when $k = 14/a$.

In conclusion the author expresses his sincere thanks to Prof. B. Sen, D. Sc., F. N. I. for his kind guidance in the preparation of this note.

DEPARTMENT OF MATHEMATICS,
JADAVPUR UNIVERSITY,
CALCUTTA-32

References

Biezeno and Grammel (1955), *Engineering Dynamics*, Vol III, p 37.
British Association, (1937), *Reports of Math. Tables Commission*.

BOOK REVIEW

Gelfond, A.O., The Solution of Equations in Integers *P. Noordhoff N. V., The Netherlands (1960), (Translated by L. F. Boron).*

This excellent booklet is an out and out elementary introduction to the theory of Diophantine equations. Diophantine equations of first and second degrees are discussed in detail in a very clear and lucid manner. A brief beautiful elementary survey of Diophantine equations of higher degree (though not complete) is also added at the end.

The booklet is based on lectures delivered at Mathematical Olympics at Lomonsov State University of Moscow. The mathematical prerequisite of the book is elementary (School) algebra. In the discussions, the properties of continued fractions have been frequently used but these properties have been deduced explicitly in this booklet. Whenever possible, attempts are made to arouse the curiosity of the reader about higher topics as the theory of algebraic numbers, that of rational approximations of irrational numbers, and the like. Some relevent theorems namely, those of Thue, Siegel and of others (which cannot be proved by elementary methods) and higher notions like those of ring, ideal, etc. whenever required are stated clearly and explained lucidly. This book will really be accessible to the more advanced High School students who have an interest in Mathematics as claimed in the foreword of the booklet. It will be also useful and of interest to amateurs and workers in the field of the theory of numbers and to teachers of Mathematics.

M. DUTTA.

CALCUTTA MATHEMATICAL SOCIETY

Report of the Council on the state of affairs of the Society for the year 1959 placed in the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the state of affairs of the Society for the year 1959, as required by the provisions of Rule 25 of the Society's constitution

The Council. The Council of the Society for the year 1959 consisting of officers and other members elected at the last Annual General meeting with the Assistant Secretary and Editorial Secretary was constituted as follows:

President

S. N. Bose,

Vice-Presidents

Ram Behari,
V. V. Narlikar,
P. C. Mahalanobis,
B. N. Prasad,
N. R. Sen,

Treasurer

S. C. Ghosh,

Secretary

P. P. Chattarji,

Editorial Secretary

P. K. Ghosh,

Other Members of the Council

G. Bandyopadhyay,
A. C. Banerji,
S. K. Basu,
U. R. Burman,
S. K. Chakraborty,
J. De (*Asst. Secretary*),
T. C. Roy,
H. M. Sengupta,
R. N. Sen,
B. R. Seth,
C. N. Srinivasiengar,
N. G. Shabde.

General. The year started in the midst of the celebrations of the Golden Jubilee of the Society. In this year, in addition to usual activities the Society has undertaken some new ventures, viz. the renovation of the Library of the Society, the organisation of Symposia on recent advances in different branches of Mathematics, the arrangement of publication of a supplementary number of the Bulletin of the Society such that no paper accepted for publication will be awaiting publication for a long time. For certain limitations, it was not possible for the Society to undertake a few more important works, viz, the publication of the short report of the progress of the researches in Mathematics in India in the last fifty years, the organisation of the symposium on teaching in Mathematics to devise means for improving the standard of Mathematics in India, the increase in the number of subscribed journals in the Society. The short report of activities of the Society in different heads is given below:

Special Lectures. On the 10th. January, 1959 at 4-30 P.M. Prof. A. C. Banerji formerly the Head of the Department of Mathematics and the Vice-Chancellor of the Allahabad University, delivered an interesting and informative lecture on "Space Travel" in connection with the Golden Jubilee Celebrations of the Society.

On the 3rd. February, 1959 at 4-30 P.M., Prof. N. R. Sen, the Head of the Department of Applied Mathematics, Calcutta University delivered an interesting and instructive lecture on "The Sun" in connection with the Golden Jubilee Celebrations of the Society. In his lecture he gave a short report of the recent progress of researches regarding the Sun.

On February 16, 1959 at 11-30 A.M., Academician L. Janossy of Hungary delivered an illuminating lecture on "Statistical Evaluation of Measurements".

The above lectures, though extremely useful for the active scientists in the field, could not be published for want of money.

In a special general meeting of the Calcutta Mathematical Society Sir H. Jeffreys and Lady Jeffreys were present and there was a learned discussion between Sir H. Jeffreys and members of the Society regaridng recent developments in some branches of Mathematics.

Symposia. On the 2nd., 3rd., 4th., of October, 1959, a symposium on "Elasticity, Plasticity and Rheology" was held according to the following programme: National Prof. S. N. Bose, F.R.S. presided over the meetings on the 2nd. and 3rd. and Prof. B. B. Sen, D.Sc. on the 4th.

Speaker:	*Topic:*
Prof. B. B. Sen	Direct Determination of Stresses.
Dr. P. D. S. Verma	Hypo-elasticity.
Dr. G. Paria	Poro-elastic media.
Dr. S. C. Das Gupta	Waves in Elastic Media.
Dr. Saktikanta Chakraborty	Disturbances in Elastic Media.
Dr. B. Karunes	Shock wave propagation Effects in Impact Problems of Plasticity.

Dr. D. N. Mitra	Application of Function-theoretic Method to Problems of Elasticity.
Dr. P. P. Chattarji	Finite Strain Theory.
Sri M. Mitra	The Source Problem in Seismology.
Sri Avtar Singh	Thermo-elastic Problems for Anisotropic Media.
Sri S. Dutt.	Effects of Cross-elasticity.

For want of funds, the lectures could not be published. The abstracts of these lectures will be published in the Bulletin of the Calcutta Mathematical Society.

The work for organisation of a symposium on Teaching of Mathematics is going ahead.

Publications. During the year under review the Society has published four issues of the Bulletin, viz. vol. 50, No. 4, and vol. 51, Nos. 1-3. The Council takes this opportunity to record here the Society's indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the officers and members of the staff of the University Press for their valued services. It may, however, be mentioned that the Society bears the cost of paper and blocks and the matter of the bulletin is composed by the Society's compositors.

The Council is glad to announce that the printing of the Golden Jubilee Commemoration Volume is in progress and is expected to be completed within a few months. The Council expresses the Society's indebtedness to the University of Calcutta for the benevolent patronage in connection with this publication.

The Council is also glad to announce that the printing of the supplement number of the Bulletin containing the arrear papers is in progress and is expected to be completed within three months. The Council expresses the Society's gratitude to the Central Government and to the Government of West Bengal for their grants which have enabled the Society to undertake publication of this supplement number.

For want of funds the Society is not at present in a position to publish the short report of the progress of researches in Mathematics in India during in the last fifty years.

Exchange of Publications. The transmission of the Society's publications to various countries of the world has been carried on regularly during the year and seven new exchange relations have also been established. Altogether one hundred and forty journals have been received in exchange of our Bulletin this year.

Library. Inspite of the hard financial position, the Society has purchased several journals in this year. In this connection, the Council expresses thanks to the American Mathematcal Society for agreeing to send the "Mathematical Reviews" regularly in exchange of our Bulletin. For want of funds, the Society is not able to purchase a few journals not available in exchange but considered very useful by local active research workers. For the same reason, the Society has not been able to purchase any book with an honourable exception of only one at the price of Rs. 10.20. Though in the collection of the Library there are some rare valuable books, yet as there is practically no new addition, the library is not able to serve the needs of the active research workers to the desired extent. The Council is glad to announce that a systematic catalogue of books and comprehensive list of journals have been prepared.

Amendment of Rule. On the recommendation of the Council the following alteration of Rule 15 of the existing Rules and Regulations of the Society was made by majority of votes of the Members of the Society.

The rule 15 of Regulations was:

> "Members who have paid the admission fee and a sum of One hundred rupees in advance shall become Life Members."

The rule as amended is:

> "Members who have paid the admission fee and a sum of Rs. 192/- (*i.e.*, sixteen times the usual Annual Subscription for ordinary membersip) in advance shall become Life Members. A rebate on the subscription for Life Membership shall be allowed to a member for each year of previous membership at the rate of $33\frac{1}{3}\%$ of the usual subscription already paid upto a maximum period of twelve years."

Meetings during 1959. The Council has met six times during this year and there have been six General Meetings including one special Meeting of members in which nine original papers have been discussed and twenty-one original papers have been taken as read. In the special general meeting Sir H. Jeffreys and Lady Jeffreys were present,

Membership. The Council desires to report that during the year under review twenty new names have been added to the list of Society's members. Two ordinary members have become Life Members of the Society.

Delegations to learned societies. In reply to invitations from Indian Science Congress Association the Council nominated the following members to represent the Society at their 47th. Session

(1) Prof. B. N. Prasad—Professor of Mathematics, Allahabad University.
(2) Dr. H. M. Sengupta—Reader in Pure Mathematics, Calcutta University.
(3) Dr. M. Dutta—Reader in Mathematics, Jadavpur University.
(4) Dr. M. C. Chaki—Lecturer in Pure Mathematics, Calcutta University.

At the 25th. session of the Indian Mathematical Society at Allahabad, the Council nominated the following members to represent the Society:

(1) Dr. A. C. Choudhury—Reader in Pure Mathematics, Calcutta University.
(2) Dr. G. Bandyopadhyay—Asst. Professor of Mathematics, Indian Institute of Technology, Kharagpur.

The Council nominated the following members to represent the Society at the Fifth session of the Indian Standards Convention at Hyderabad:

Prof. B. R. Seth—Professor of Applied Mathematics, Indian Institute of Technology, Kharagpur.

Finance. The annual accounts of the Society have been presented to the Council in the standardized form by the auditors Sarbashree B. C. Chatterjee and S. K. Chakraborty. The Council offers them its sincere thanks for their honorary services. The Council also takes this opportunity to make a few observations on the audited statement of accounts during the year under review. A glance at the receipt and expenditure sides of the statement at once reveals an extremely gloomy picture regarding the financial position of the Society. Though we received usual grants during the year under review from the Government of West Bengal, we have received a grant from the Government of India only for arrear publication.

No doubt there has been some increase in the receipts due to the enhancement of the price of the back volumes of the Bulletin, but almost the entire amount has been exhausted to cope with the day-to-day increasing market price. The Council is of considered opinion that the financial position of the Society is very precarious. Unless the Society receives liberal grants-in-aid from Government and donations from members and public, it will be very difficult for the Society to pursue its aim and objects in the desired manner.

The Society received the following grants during the year under review:

(i)	Government of West Bengal	Rs.	2,000/-
(ii)	National Institute of Sciences of India	,,	1,000/-
(iii)	Government of India (for Arrear publication)	,,	2,000/-
(iv)	do do (Symposium on Teaching)	,,	2,000/-

The Council offers its grateful thanks to the Govt. of West Bengal, the Govt. of India and the National Institute of Sciences of India for these grants.

The Society received the following grants and donations for the celebration of its Golden Jubilee:

1. Grants:—

(i)	Government of India	... Rs.	2,500/-
(ii)	Government of West Bengal	... Rs.	4,000/-
(iii)	Government of Assam	... Rs.	500/-
(iv)	University of Calcutta	... Rs.	1,000/-

2. Donations:—

(i)	Members of the Jubilee Committee and Society	... Rs.	3,288/50
(ii)	Others (Institutions and individuals)	... Rs.	1,518/20

The Council offers its thanks to the Government of India, the Government of West Bengal, the Government of Assam, the University of Calcutta, the members of the Jubilee Committee and Society and others (institutions and individuals) for the grants and donations which have made the celebrations of the Golden Jubilee a success.

All correspondence with the Society, subscriptions to the Bulletin, admission fees and annual contributions of members are to be sent to the *Secretary, Calcutta Mathematical Society, 92 Upper Circular Road, Calcutta-9.*

Papers intended for publication in the Bulletin of the Society, and all Editorial Correspondence, should be addressed to the *Editorial Secretary, Calcutta Mathematical Society.*

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office, or from its agents—Messrs. Bowes & Bowes, Booksellers and Publishers, 1, Trinity Street, Cambridge, England.

NOTICE TO AUTHORS

The manuscript of each paper communicated for publication in this Journal should be legibly written (preferably type-written) on one side of the paper and should be accompanied by a short *abstract* of the paper.

References to literature in the text should be given, whenever possible, in chronological order, only the names of the authors and years of publication, in brackets, being given. They should be cited in full at the end of paper, the author's name following form, ***viz.***, name or names of authors; year of publication; name of the journal (abbreviation); number of volume; and lastly, the page number. The following would be a useful illustration:

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), **21**, 285.

Authors of papers printed in the Bulletin are entitled to receive, free of cost, 50 separate copies of their communications. They can, however, by previous notice to the Secretary, ascertain whether it will be possible to obtain more copies even on payment of the usual charges.

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1960

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CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST. DECEMBER, 1959.

Receipts.

Opening Balance		Rs. nP.	Rs. nP.
(a) With Secretary			
(i) In cash	4.27		
(ii) In stamps	0.54		
		4.81	
(b) Balance at Banks			
(i) State Bank of India (Gen. Fund)	459.30		
(ii) Do (K. K. G. P. Fund) ...	992.60		
(iii) Do (C. E. Cullis Fund) ...	717.00		
(iv) United Bank of India	475.13		
Do (in suspense)	44.58		
(v) Postal Savings Bank	35 50		
		2,724.11	
(c) G. P. Notes (General Fund) (Face value Rs 6,000.00)		5,663.72	
(d) G. P. Notes (K. K. G. P Fund) (Face value Rs. 2,000.00)		1,937.48	
(e) G. P. Notes (C. E. Cullis Fund) (Face value Rs. 8,000.00)		6,558.06	
			16,888.18
2. Membership subscription (including Life Membership subscription 200.00) ...	1,674.62		
3. Admission fees	200.00		
4. Sale proceeds	3,883.61		
5. Donation	Nil		
		5,758.23	
6. Receipt on Jubilee Celebration Fund		23 00	
7. Grants			
(i) Govt. of West Bengal	2,000.00		
(ii) Nat. Inst. of Sciences.	1,000.00		
(iii) Govt. of India (Arrear)	2,000.00		
(iv) Do (Symposium Teaching) ...	2,000.00		
		7,000.00	
7. Interest			
(a) G. P.Notes (General Fund) ...	180.00		
(b) Do (K. K. G. P. Fund)	60.00		
(c) Do (C. E Cullis Fund)	240.00		
(d) Postal Savings Accounts	Nil		
		480.00	
			13,261.23
		Total ... Rs.	30,149.41

Disbursements

		Rs. nP.	Rs. nP.
1. Establishment		377 71	
2. Meetings		280.53	
3. Books & Journals (including binding)			
(a) Books	10 20		
(b) Journals	1,821.20		
(c) Binding	483.50		
		2,314.90	
4. Bulletins			
(a) Establishment	921 27		
(b) Compositors' Salary	2,202.00		
(c) Papers, Blocks, Types etc	1,217.62		
(d) Postage	569.97		
(e) Conveyance (Press)	27.99		
		4,938.85*	
5. Printing & Stationery		136.01	
6. Postage (General)		245.62	
7. Bank charges		37.33	
8. Miscellaneous (including conveyance charges)		117 34	
9. Payments to Jubilee Celebration Fund accounts		23.00	
10. Symposium (General)		83.70	
			8 554.99
11. Symposium on Teaching			389.29**
			8,944.28
9. Closing Balance			
(a) With Secretary			
(i) In cash	4.31		
(ii) In stamps	1.25		
	5.56		
(b) Balance at Banks			
(i) State Bank of India (Gen. Fund)	2,704 80		
(ii) Do (K. K. G P. Fund) ...	1,052 10		
(iii) Do C. E. Cullis Fund) ...	956 00		
(iv) United Bank of India	2,077.90		
Do (in suspense)	214.01		
(v) Postal Savings	35.50		
		7,045 87	
(c) G. P Notes (General Fund) (Face value Rs. 6,000.00)		5,663 72	
(d) Do (K. K G P. Fund) (Face value Re 2,000 00)		1,937 48	
(e) (C. E. Cullis Fund) Face value Rs. 8,000.00)		6,558.06	
			21,205.1
		Total ... Rs.	30,149 4

To

The Members of the Calcutta Mathematical Society,

We have examined the above accounts with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

* Further commitments of Rs 2,500/00

** Further commitments of Rs 1,600/00

The actual expenditure over the income during the year including commitments is Rs. 13,014.23 instead of Rs. 8,991.28 as shown here.

B. C. Chatterjee } *Auditors*
S. K. Chakraborty

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ON THE SLOW STEADY ROTATION OF A SPHEROID OF SMALL ELLIPTICITY IN A VISCOUS LIQUID

By

S. R. Khamrui, *Jadavpur, Calcutta*

(*Received—February 15, 1960.*)

In the case of slow steady motion of a viscous liquid due to the rotation of a sphere about a diameter, a solution was constructed by the author (Khamrui, 1956) by a process of successive approximations, which shows that the motion consists of an inflow at the poles and an outflow at the equator. In this paper the steady rotation of a spheroid of small ellipticity in a viscous liquid is studied and adopting the same method a solution is constructed which gives a picture similar to that of the sphere.

1. Using the spherical polar co-ordinates (r, θ, φ), let us suppose that the spheroid

$$r = a\{1+\epsilon P_2(\cos\theta)\}$$

where $P_2(\cos\theta)$ is Legendre's polynomial of degree 2 in $\cos\theta$ and ϵ is very small so that its square and higher powers may be neglected, rotates slowly with angular velocity Ω about the polar axis. If u, v, w be the components of velocity, then omitting the azimuthal variations, the equations of steady motion and the equation of continuity are (Goldstein, 1950).

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\theta}-\frac{v^2+w^2}{r}=-\frac{1}{\varrho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 u-\frac{2u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\theta}-\frac{2v\cot\theta}{r^2}\right) \tag{1}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\theta}+\frac{uv}{r}-\frac{w^2\cot\theta}{r}=-\frac{1}{\varrho}\frac{1}{r}\frac{\partial p}{\partial\theta}+\nu\left(\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial\theta}-\frac{v}{r^2\sin^2\theta}\right) \tag{2}$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\theta}+\frac{wu}{r}+\frac{vw\cot\theta}{r}=\nu\left(\nabla^2 w-\frac{w}{r^2\sin^2\theta}\right) \tag{3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta)=0. \tag{4}$$

If we assume $p =$ contant, $u = v = 0$ and neglect the 2nd order terms in w, we get

$$\nabla^2 w-\frac{w}{r^2\sin^2\theta}=0. \tag{5}$$

Since the velocity must vanish at infinity we can take the solution of (5) in the form (Jeffery, 1915)

$$w=\sum_{n=1}^{\infty}\frac{A_n}{r^{n+1}}P_n^1(\cos\theta) \tag{6}$$

where $P^1_n(\cos\theta)$ is the associated Legendre function of the first kind, of degree n and of the first order.

The boundary condition is

$$w = \Omega r \sin\theta \qquad \text{when} \quad r = a\{1+\varepsilon P_2(\cos\theta)\} \tag{7}$$

We note the following well known properties of Legerdre's Coefficients:

$$P^1{}_n(\mu) = (1-\mu^2)^{\frac{1}{2}} \frac{dP_n(\mu)}{d\mu}$$

$$\frac{dP_n(\mu)}{d\mu} = (2n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu) + (2n-9)P_{n-5}(\mu) + \dots$$

the last term being $P_0(\mu)$ or $3P_1(\mu)$ according as n is odd or even.

$$P_2(\mu)\, P^1{}_n = (1-\mu^2)^{\frac{1}{2}}\left[\frac{3}{2}\frac{n(n+1)}{2n-1}\{P_{n+1}(\mu) - P_{n-1}(\mu)\} + \{(2n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu)) + \dots\}\right]$$

Neglecting the second and higher powers of ε, we have when

$$r = a\{1+\varepsilon P_2(\cos\theta)\}$$

$$\frac{1}{r^{n+1}} = \frac{1}{a^{n+1}}\{1-(n+1)\varepsilon P_2(\cos\theta)\}.$$

Therefore on writing μ for $\cos\theta$, we get from (7),

$$\Omega a\{1+\varepsilon P_2(\mu)\} = \sum_{n=1}^{\infty}\frac{A_n}{a^{n+1}}\Bigg[\Big\{(2n-1)P_{n-1}(\mu) + (2n-5)P_{n-3}(\mu) + \dots$$

$$-\varepsilon.(n+1)\Big(\frac{3}{2}.\frac{n(n+1)}{2n+1}\{P_{n+1}(\mu) - P_{n-1}(\mu)\} + \{(2n-1)P_{n-1}(\mu)$$

$$+ (2n-5)P_{n-3}(\mu) + \dots\Big\}\Big)\Bigg]. \tag{8}$$

Equating the cofficients of $P_0(u)$, $P_1(\mu)$, $P_2(\mu)$...the following set of equations are obtained to determiene the constants:

(i) $$\Omega a^2 = \frac{A_1}{a} + \frac{A_3}{a^3}(1-4\varepsilon) + \frac{A_5}{a^5}(1-6\varepsilon) + \dots$$

(ii) $$0 = \frac{A_2}{a^2}\left(3 - \frac{18\varepsilon}{5}\right) + \frac{A_4}{a^4}(3-15\varepsilon) + \frac{A_6}{a^6}(3-21\varepsilon) +$$

(iii) $$\Omega a^7\varepsilon = \frac{A_1}{a}(-2\varepsilon) + \frac{A_3}{a^3}\left(-\frac{18}{7}\varepsilon\right) + \frac{A_5}{a^5}(5-30\varepsilon) + \dots$$

(iv) $$0 = \frac{A_2}{a^2}\left(-\frac{27\varepsilon}{5}\right) + \frac{A_4}{a^4}\left(7-\frac{55}{3}\varepsilon\right) + \frac{A_6}{a^6}(7-49\varepsilon) + \dots$$

(v) $$0 = \frac{A_3}{a^3}\left(-\frac{72}{7}\varepsilon\right) + \frac{A_5}{a^5}\left(9-\frac{324}{11}\varepsilon\right) + \frac{A_7}{a^7}(9-72\varepsilon) + \dots$$

etc.

We have taken ε to be very small and have retained only its first power. For a sphere, we have A_2, A_3, A_4...all zero, therefore for the case of the spheroid the values of A_2, A_3, A_4...will contain ε as factor; hence we neglect terms like εA_2, εA_3 . as we take into account only the first power of ε. Thus from (*ii*), (*iv*) and other subsequent equations with even numbers, it will be found that $A_2 = A_4 = A_6 = \ldots = 0$ *i.e.* A's with even suffixes are all zero. From (*i*), (*ii*), (*v*) and other subsequent equations with odd numbers it is clear that A_5, A_7...will contain ε^2, ε^3...as factors and consequently they are neglected. Thus we get

$$A_1 = \Omega a^3 \frac{5-\varepsilon}{5+2\varepsilon} = \Omega a^3\left(1-\frac{3\varepsilon}{5}\right)$$

$$A_3 = \frac{3\Omega a^5\varepsilon}{5},$$

$$\therefore \quad w = \frac{\Omega a^3}{r^2}\left(1-\frac{3\varepsilon}{5}\right)P^1{}_1(\cos\theta)+\tfrac{3}{5}\frac{\Omega a^5\varepsilon}{r^4}P^1{}_3(\cos\theta)$$

$$= \frac{\Omega a^3}{r^2}\left(1-\frac{3\varepsilon}{5}\right)\sin\theta+\tfrac{3}{5}\cdot\frac{\Omega a^5\varepsilon}{r^4}\{5P_2(\cos\theta)+1\}\sin\theta.$$

The only strenss components that contributes to the retarding couple is

$$\widehat{r\varphi} = \mu\left(\frac{1}{r\sin\theta}\,\varepsilon\frac{\partial u}{\varphi}+\frac{\partial w}{\partial r}-\frac{w}{r}\right)$$

$$= \mu\left(\frac{\partial w}{\partial r}-\frac{w}{r}\right).$$

The retarding couple is found to be

$$M = 24\pi\mu\Omega a^3(\tfrac{1}{3}-\tfrac{1}{5}\varepsilon).$$

Hence the first approximate solution of the problem is

$$\left.\begin{aligned} u = v = 0,\quad w &= \frac{\Omega a^3}{r^2}\left(1-\frac{3\varepsilon}{5}\right)\sin\theta+\tfrac{3}{5}\cdot\frac{\Omega a^5\varepsilon}{r^4}\{5P_2(\cos\theta)+1\}\sin\theta \\ p = \text{constant},\quad M &= 24\pi\mu\Omega a^3(\tfrac{1}{3}-\tfrac{1}{5}\varepsilon). \end{aligned}\right\} \qquad (9)$$

In we put $\varepsilon = 0$, we get the results for the sphere of radius a.

2. In the second approximation, we take the components of velocity as

$$u, v, w = \frac{\Omega a^3}{r^2}\left(1-\frac{3\varepsilon}{5}\right)\sin\theta+\tfrac{3}{3}\frac{\Omega a^5\varepsilon}{r^4}\{5P_2(\cos\theta)+1\}\sin\theta+w'.$$

Substituting in the left hand sides of the equations (1)-(3) from the first approximation (9), we get

$$\nu\left[\nabla^2 u-\frac{2u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\theta}-\frac{2v\cot\theta}{r^2}\right]=\frac{1}{\varrho}\frac{\partial p}{\partial r}-\left\{\frac{\Omega^2a^6}{r^5}\left(1-\frac{6\varepsilon}{5}\right)+\frac{36}{5}\cdot\frac{\Omega a^8\varepsilon}{r^7}\right\}\sin^2\theta+\frac{9\Omega^2a^8\varepsilon}{r^7}\sin^4\theta \tag{10}$$

$$\nu\left[\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial\theta}-\frac{v}{r^2\sin^2\theta}\right]=\frac{1}{\varrho}\frac{\partial p}{r\partial\theta}-\left\{\frac{\Omega^2a^6}{r^5}\left(1-\frac{6\varepsilon}{5}\right)+\frac{36}{5}\cdot\frac{\Omega^2a^3}{r^7}\right\}\sin\theta\cos\theta+\frac{9\Omega^2a^8\varepsilon}{r^7}\sin^3\theta\cos\theta \tag{11}$$

$$\nabla^2 w'-\frac{w'}{r^2\sin^2\theta}=0. \tag{12}$$

Since w' is a solution of (12) and will vanish at infinity as well as on the boundary $r=a\{1+\varepsilon P_2(\cos\theta)\}$, therefore $w'=0$.

Let us take

$$u=\Omega^2a^2[F_1(r)+F_2(r)\sin^2\theta+F_3(r)\sin^4\theta] \tag{13}$$

$$v=\Omega^2a^2\sin\theta\cos\theta[G_1(r)+G_2(r)\sin^2\theta] \tag{14}$$

$$p=2\varrho\nu\Omega^2a^2[f_1(r)+f_2(r)\sin^2\theta+f_3(r)\sin^4\theta]. \tag{15}$$

Substituting in (4), (10) and (11), we get

$$F'_1+\frac{2}{r}F_1+\frac{2}{r}G_1=0 \tag{16}$$

$$F'_2+\frac{2}{r}F_2-\frac{3G_1}{r}+\frac{4G_2}{r}=0 \tag{17}$$

$$F'_3+\frac{2}{r}F_3-\frac{5}{r}G_2=0 \tag{18}$$

$$F''_1+\frac{4}{r}F'_1+\frac{2}{r^2}F_1+\frac{4}{r^2}F_2=2f'_1 \tag{19}$$

$$F''_2+\frac{4}{r}F'_2-\frac{4}{r^2}F_2+\frac{16}{r^2}F_3=2f'_2-\frac{a^4}{\nu r^5}\left(1-\frac{6\varepsilon}{5}\right)-\frac{36}{5}\cdot\frac{a^6\varepsilon}{\nu r^7} \tag{20}$$

$$F''_3+\frac{4}{r}F'_3-\frac{18}{r^2}F_3=2f'_3+\frac{9a^6\varepsilon}{\nu r^7} \tag{21}$$

$$G''_1+\frac{2}{r}G'_1-\frac{6G_1}{r^2}+\frac{8G_2}{r^2}+\frac{4F_2}{r^2}=\frac{4f_2}{r}-\frac{a^4}{\nu r^5}\left(1-\frac{6\varepsilon}{5}\right)-\frac{36}{5}\cdot\frac{a^6\varepsilon}{\nu r^7} \tag{22}$$

$$G''_2+\frac{2}{r}G'_2-\frac{20}{r^2}G_2+\frac{8F_3}{r^3}=\frac{8f_3}{r}+\frac{9a^6\varepsilon}{\nu r^7} \tag{23}$$

where dash denotes differntiation with respect to r.

The conditions to be satisfied are

(*i*) $u=v=0$ when $r=a\{1+\varepsilon P_2(\cos\theta)\}$

(*ii*) u, v must vanish at infinity and pressure must be finite there.

From the above equations and the boundary conditions, the functions are determined od substituting their values in (13), (14), (15) and rearranging we get

$$= \Omega^2 a^2\Big[-\frac{a^3}{8\nu r^2}\Big(1-\frac{a}{r}\Big)^2(3\cos^2\theta-1)-\frac{a^3\varepsilon}{\nu r^2}\Big\{\Big(\frac{127}{140}+\frac{8}{5}\cdot\frac{a}{r}-\frac{499}{140}\frac{a^2}{r^2}+\frac{6}{5}\frac{a^3}{r^3}+\frac{6}{7}\frac{a^4}{r^4}\Big)$$

$$-\Big(\frac{1389}{280}+\frac{9}{10}\frac{a}{r}-\frac{1077}{280}\frac{a^2}{r^2}-\frac{63}{10}\frac{a^3}{r^3}+\frac{30}{7}\frac{a^4}{r^4}\Big)\sin^2\theta+\Big(\frac{15}{8}\frac{a^2}{r^2}-\frac{45}{8}\frac{a^3}{r^3}+\frac{15}{4}\frac{a^4}{r^4}\Big)\sin^4\theta\Big\}\Big] \quad (24)$$

$$= \Omega^2 a^3\cos\theta\sin\theta\Big[\frac{a^4}{4\nu r^3}\Big(1-\frac{a}{r}\Big)-\frac{a^4\varepsilon}{\nu r^3}\Big\{\Big(\frac{3}{10}-\frac{499}{140}\frac{a}{r}+\frac{9}{5}\frac{a^2}{r^2}+\frac{12}{7}\frac{a^3}{r^3}\Big)$$

$$-\Big(\frac{3}{4}\frac{a}{r}-\frac{27}{8}\frac{a^2}{r^2}+3.\frac{a^3}{r^3}\Big)\sin^2\theta\Big\}\Big] \quad (25)$$

$$p = 2\varrho\nu\Omega^2 a^2\Big[c-\frac{a^3}{4\nu r^3}\Big(1-\frac{a}{r}\Big)+\frac{3a^3}{8\nu r^3}\Big(1-\frac{4}{3}\cdot\frac{a}{r}\Big)\sin^2\theta-\frac{a^3\varepsilon}{\nu r^3}\Big\{\Big(\frac{463}{140}+\frac{8}{10}\frac{a}{r}$$

$$+\frac{3}{5}\frac{a^2}{r^2}-\frac{33}{10}\frac{a^3}{r^3}\Big)-\Big(\frac{1389}{280}-\frac{8}{5}\cdot\frac{a}{r}+\frac{3a^2}{r^2}-\frac{351}{20}\cdot\frac{a^3}{r^3}\Big)\sin^2\theta+\frac{3}{2}\frac{a^2}{r^2}\Big(\frac{7}{4}-3.\frac{a}{r}\Big)\sin^4\theta\Big\}\Big] \quad (26)$$

where c is a constant.

8. Since ε is small, it is found that when $\theta = 0$, $v = w = 0$ and u is negative so that there is purely radial inflow at the poles. When $\theta = 90°$, $v = 0$, so that the outflow is in the equatorial plane, but not purely radial.

As u is independent of φ and $w' = 0$, the retarding couple is the same as in the first approximation.

Putting $\varepsilon = 0$, we get the results for the rotation of the sphere of radius a which agree with author's results.

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ON LAGUERRE TRANSFORM

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1. **Introduction.** Churchill (1954) defined the Legendre transform $f(n)$ of $F(x)$, when $f(n)$ and $F(x)$ are connected by means of the integral

$$f(n) = \int_{-1}^{1} F(x)\, P_n(x)\, dx, \quad (n = 0, 1, 2, \ldots\ldots)$$

where $P_n(x)$ is the Legendre Polynomial of degree n. The transform was utilised in solving some boundary value problem.

The object of this paper is to introduce Laguerre transform and to study some of its properties.

2. **Definition and Properties of Laguerre Transform.** The Laguerre transform of $F(x)$, denoted by $f_\alpha(x)$ is defined by

$$f_\alpha(n) = T\{F(x)\} = \int_{0}^{\infty} e^{-x} x^\alpha L_n^\alpha(x)\, F(x)\, dx \tag{1}$$

where the integral on the right of (1) exists in the sense of Lebesgue or in particular $F(x)$ is integrable (L) in $(0, \infty)$; $L_n^\alpha(x)$ being the Laguerre polynomial of degree n. (n non-negative integer; $\alpha > -1$). The Laguerre polynomial $L_n^\alpha(x)$ is a solution of the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \tag{2}$$

$$y = L_n^\alpha(x)$$

The self adjoint differential form of $L_n^\alpha(x)$ is given by

$$\frac{d}{dx}\left[e^{-x} x^{\alpha+1} \frac{d}{dx}(x)\right] + n\, e^{-x} x^\alpha L_n^\alpha(x) = 0 \tag{3}$$

We have the orthogonal relations and for Laguerre Polynomials

$$\left.\begin{aligned} \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x)\; L_m^\alpha(x)\, dx &= 0 \quad n \neq m \\ &= \delta_n \quad n = m \end{aligned}\right\} \tag{4}$$

where

$$\delta_n = \Gamma(\alpha + 1) \binom{n + \alpha}{n} \tag{5}$$

Let $R[F]$ denote the differential form

$$R[F(x)] = e^{x} x^{-\alpha} \frac{d}{dx} \left[e^{-x} x^{\alpha+1} \frac{d}{dx} F(x) \right]$$

$$T\{R[F]\} = \int_0^\infty e^{-x} x^{\alpha} L_n^{\alpha}(x)\; R[F(x)]\, dx$$

$$= \int_0^\infty L_n^{\alpha}(x) \frac{d}{dx} \left[e^{-x} x^{\alpha+1} \frac{d}{dx} F(x) \right] dx$$

Integrating by parts and utilising the relation (3)

$$T\{R[F]\} = -n \int_0^\infty e^{-x} x^{\alpha} L_n^{\alpha}(x)\, F(x)\, dx$$

$$= -n\, f_\alpha(n)$$

and we have the following theorem.

Theorem 1. *If $F'(x)$ be continuous, $F''(x)$ bounded and integrable (L) in each of the intervals $0 < x < N$, $N < \infty$ and*

$$T\{F(x)\} \textit{ exists,}$$

then $T\{R[F(x)]$ exists and

$$T\{R[F(x)]\} = -n\, f_\alpha(n). \tag{6}$$

If $R[F(x)$ and $F(x)$ satisfy the conditions of the theorem 1. then

$$T\{R^2[F]\} = T\{R[R[F]]\} = n^2 f_\alpha(n)$$

$$T\{R^3[F]\} = -n^3 f_\alpha(n) = (-1)^3 n^3 f_\alpha(n).$$

and so on. Generally, if $R^k[F(x)]$ and $F(x)$ satisfy the conditions of the Theorem 1, $k = 1, 2\ldots\ldots m-1$

$$T\{R^m[F(x)]\} = (-1)^m n^m f_\alpha(n).$$

where m is any positive integer.

Let us now consider the transform of the function $R^{-1}[F(x)]$ where R^{-1} is the nverse of the differential operator R

Put $$R^{-1}[F(x)] = Y(x)$$

Then $Y(x)$ is a solution of the differential equation

$$R[Y(x)] = F(x) \tag{7}$$

Let $F(x)$ possess an indefinite Lebesgue integral and let further

$$\int_0^\infty e^{-x} x^{\alpha} F(x)\, dx = 0$$

so that $$f_\alpha(0) = 0.$$

The first integral of the equation (7)

$$e^{-x}x^{a+1}Y'(x) = \int_0^x e^{-t}t^a F(t)\,dt.$$

is a continuous function of x, with limit zero as $x \to 0, \infty$.

The second integral

$$Y(x) = \int_0^x \frac{e^s}{s^{a+1}} \int_0^s e^{-t}t^a F(t)\,dt\,ds + C = R^{-1}[F(x)]$$

where C is an arbitrary constant, is also continuous.

Evidently, $e^{-x}x^{a+1}Y(x)$ vanishes as $x \to 0, \infty$

if $| Y(x) | = o(e^x x^{-a-1})$ as $x \to 0, \infty$.

Then $T\{Y(x)\}$ exists and

$$T\{R[Y]\} = -n\,T\{Y(x)\}$$
$$= f_a(n).$$

$$T\{R^{-1}[F(x)]\} = -\frac{f_a(n)}{n}, \quad n \text{ is a positive integer.}$$

We thus obtain the following theorem.

Theorem 2. *If $F(x)$ be bounded and integrable in each of the sub-intervals of $0 < x < N$ and $f_a(0) = 0$, then $f_a(n)$ exists and for each constant C*

$$T^{-1}\left\{-\frac{f_a(n)}{n}\right\} = R^{-1}[F(x)] = \int_0^x \frac{e^s}{s^{a+1}} \int_0^s e^{-t}t^a F(t)\,dt\,ds + C$$

(n is a positive integer).

If $T\{F_1(x)\}$ and $T\{F_2(x)\}$ exists, then

$$T\{C_1 F_1(x) + C_2 F_2(x)\} = C_1 T\{F_1(x)\} + C_2 T\{F_2(x)\}$$

where C_1, C_2 are constants and the transformation $T\{F(x)\}$ is linear.

Assuming further that $F(x)$ is even, we have, the following

$$T\{F(-x)\} = \int_0^\infty e^{-x} x^a L_n^a(x) F(-x)\,dx$$
$$= f_a(n).$$

Let $$G(x) = \int_0^x F(t)\,dt$$

so that $G(x)$ is absolutely continuous and $G'(x)$ exists and let it be bounded everywhere and integrable.

Then
$$f_\alpha(n) = \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x)\, G'(x)\, dx$$
$$= \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x)\, G(x)\, dx - \alpha \int_0^\infty e^{-x} x^{\alpha-1} L_n^\alpha(x) G(x) dx$$
$$- \int_0^\infty e^{-x} x^\alpha \left\{ \frac{d}{dx} L_n^\alpha(x) \right\} G(x)\, dx.$$

Therefore
$$f_\alpha(n) - f_\alpha(n+1)$$
$$= \int_0^\infty e^{-x} x^\alpha \left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] G(x)\, dx + \alpha \int_0^\infty e^{-x} x^{\alpha-1} \left[L_{n+1}^\alpha(x) - L_n^\alpha(x)\right] G(x) dx$$
$$- \int_0^\infty e^{-x} x^\alpha \frac{d}{dx} \left[L_n^\alpha(x) - L_{n+1}^\alpha(x)\right] G(x) dx$$
$$= \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x)\, G(x)\, dx - \int_0^\infty e^{-x} x^\alpha L_{n+1}^\alpha(x)\, G(x)\, dx + \alpha \int_0^\infty e^{-x} x^{\alpha-1} L_{n+1}^{\alpha-1}(x) G(x) dx$$
$$- \int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) G(x) dx$$

so that
$$f_\alpha(n) - f_\alpha(n+1) = \alpha\, g_{\alpha-1}(n+1) - g_\alpha(n+1)$$

on using the recurrence relations for $L_n^\alpha(x)$.

Replacing n by $n-1$ and putting $\alpha = 0$ we have in particular
$$f(n) - f(n-1) = g(n)$$

or
$$T\left\{ \int_0^x F(t)\, dt \right\} = f(n) - f(n-1)$$

where
$$f(n) = T\{F(x)\} = \int_0^\infty e^{-x} L_n(x) dx$$

The above may be stated in the form of the following theorem.

Theorem 3. *If $G(x) = \int_0^x F(t)dt$ so that $G(x)$ is absolutely continuous and $G'(x)$ exists and let it bounded and integrable, then*
$$f_\alpha(n) - f_\alpha(n-1) = g_\alpha(n) - \alpha g_{\alpha-1}(n)$$

---

* If $G'(x)$ exists and is bounded everywhere, then
$$\int_a^x G'(x)dx = G(x) - G(a).$$

and
$$T\{\int_0^x F(t)dt\} = f(n)-f(n-1).$$

3. *Examples.* (i) Let $F(x)$ be any polynomial of degree m then

$$f_\alpha(n) = 0 \qquad \text{for} \qquad n > m.$$

(ii) Let $F(x) = \overset{\alpha}{L}_m(x)$

Then
$$T\{\overset{\alpha}{L}_m(x)\} = \int_0^\infty e^{-x}x^\alpha \overset{\alpha}{L}_n(x)\overset{\alpha}{L}_m dx = 0, \quad n \neq m$$

$$= \delta_n, \quad n = m$$

(iii) Let $F(x) = x^{s-1}$ $\quad s$, real (>0)

Then
$$T\{x^{s-1}\} = \int_0^\infty e^{-x}\overset{\alpha}{L}_n(x).x^{\alpha+s-1}dx \qquad \text{[Howell (1938)]}$$

$$= \frac{\Gamma(s+\alpha)\Gamma(n-s+1)}{n\,!\,\Gamma(1-s)}$$

In particular,
$$T\{1/s\} = \Gamma(\alpha) \qquad \alpha > 0$$

(iv) Let $F(x) = e^{kx}$ where $\mathbf{Re}(k) < 1$

Then
$$T\{e^{kx}\} = \int_0^\infty e^{-x(1-k)}x^\alpha \overset{\alpha}{L}_n(x)dx$$

$$= \frac{\Gamma(n+\alpha+1)(-k)^n}{n\,!\,(1-k)^{n+\alpha+1}}$$

(Erdélyi (1954), p. 174)

(v) Let $F(x) = e^{-px}\overset{\alpha}{L}_m(x)$, $\quad \mathbf{Re}\ (p) > -1.$

[Howell (1938)]

$$T\{e^{-px}\overset{\alpha}{L}_m(x)\} = \int_0^\infty e^{-x(p+1)}x^\alpha \overset{\alpha}{L}_n(x)\overset{\alpha}{L}_m(x)dx$$

$$= \frac{1}{n\,!\,m\,!}\ \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)}{\Gamma(1+\alpha)} \cdot \frac{(p-1)^{n-m+\alpha+1}}{p^{n+m+2\alpha+2}} \quad \times {}_2F_1\left(n+\alpha+1, \quad \begin{matrix} m+\alpha+1 \\ \alpha+1 \end{matrix} \ \frac{1}{p^2}\right).$$

(vi) We have the relation ; (Erdélyi (1953), p. 192)).

$$\overset{\alpha}{L}_n(x) = \sum_{m=0}^{n} (m\,!)^{-1}(\alpha-\beta)_m \overset{\beta}{L}_{n-m}(x). \tag{8}$$

Then
$$T\{F(x).x^{\beta-\alpha}\} = \int_0^\infty e^{-x} x^{\beta} \overset{\alpha}{L}_n(x) F(x)\,dx.$$
$$= \int_0^\infty e^{-x} x^{\beta} \sum_{m=0}^{n} (m\,!)^{-1} (\alpha-\beta)_m \overset{\beta}{L}_{n-m}(x) F(x)\,dx$$
$$= \sum_{m=0}^{n} (m\,!)^{-1} (\alpha-\beta)_m f_\beta(n-m).$$

In particular when
$$\beta = \alpha - 1.$$
$$T\left\{\frac{F(x)}{x}\right\} = \sum_{m=0}^{n} f_{\alpha-1}(m).$$

(vii) We have,
$$\Gamma(\alpha, x) e^x x^{-\alpha} = \sum_{n=0}^{\infty} (n+1)^{-1} \overset{\alpha}{L}_n(x), ;\ (\alpha > -1,\ x > 0)$$

(Erdélyi (1953) p. 25)

so that
$$T\{\Gamma(\alpha, x) e^x x^{-\alpha}\}$$
$$= \int_0^\infty e^{-x} x^{\alpha} \left\{ \sum_{n=0}^{\infty} (n+1)^{-1} \overset{\alpha}{L}_n(x) \right\} \overset{\alpha}{L}_n(x)\,dx$$
$$= \sum_{n=0}^{\infty} (n+1)^{-1} \delta_n. \quad (-1 < \alpha < 0)$$

(viii) We have
$$x^{\rho} = \Gamma(\alpha+\rho+1) \sum_{n=0}^{\infty} \frac{(-\rho)_n}{\Gamma(n+\alpha+1)} \overset{\alpha}{L}_n(x),$$
$$\left[-\rho < 1 + \min\left(\alpha, \frac{\alpha}{2} - \frac{1}{4}\right),\ x > 0,\ \alpha > -1\right]$$

[Erdélyi (1953), p. 214]

so that
$$T\{x^{\rho}\} = \int_0^\infty e^{-x} x^{\alpha} \overset{\alpha}{L}_n(x) x^{\rho}\,dx$$
$$= \Gamma(\alpha+\rho+1) \sum_{n=0}^{\infty} \frac{(-\rho)_n}{\Gamma(\alpha+n+1)} \delta_n \qquad (\rho > 0)$$

(ix) From the generating functions (Erdélyi (1953), p. 189)
$$(1-z)^{-\alpha-1} e^{xz/z-1} = \sum_{n=0}^{\infty} \overset{\alpha}{L}_n(x) z^n, \qquad |z| < 1$$
$$(xz)^{-\alpha/2} e^z J_\alpha[2(xz)^{\frac{1}{2}}] = \sum_{n=0}^{\infty} \frac{L_n(x)}{\Gamma(n+\alpha+1)} z^n.$$

We obtain $$T\{(1-z)^{-\alpha-1}e^{xz/z-1}\} = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-x}x^{\alpha}L_n^{\alpha}(x)L_n^{\alpha}(x)\,dx.$$

$$= \sum_{n=0}^{\infty} \delta_n z^n \qquad |z|<1, \quad \alpha \geqq 0$$

$$T\{(xz)^{-\frac{1}{2}\alpha}e^{z}J_{\alpha}[2(xz)^{\frac{1}{2}}]\} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha+n+1)} \int_0^{\infty} e^{-x}x^{\alpha}L_n^{\alpha}(x)L_n^{\alpha}(x)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{\delta_n}{\Gamma(\alpha+n+1)}\,.z^n,$$

for every $z (\alpha > -1)$

(x) We have $$L_m^{\alpha}(x).L_n^{\alpha}(x) = \sum_{p=0}^{2\min(m,n)} l_p\cdot L_{m+n-p}^{\alpha}(x)$$ [Erdélyi (1938), p. 154]

where $$l_p = (-)^p\frac{(m+n-p)!\,\Gamma(\alpha+m+1)\Gamma(\alpha+n+1)}{p!\,\Gamma(\alpha+m+n-p+1)} \times \sum \frac{(2r)!}{r!\,(m-r)!\,(2r-p)!\,\Gamma(\alpha+r+1)}$$

$$\tfrac{1}{2}p \leq r \leq \min(m, n)$$

$$T\{L_m^{\alpha}(x)\} = \int_0^{\infty} e^{-x}x^{\alpha}L_n^{\alpha}(x).L_m^{\alpha}(x)\,dx$$

$$= \sum_{p=0}^{2\min(m,n)} l_p\cdot \int_0^{\infty} e^{-x}x^{\alpha}L_{m+n-p}^{\alpha}(x)\,dx$$

$$= \sum_{p=0}^{2\min(m,n)} l_p f_{\alpha}(m+n-p).$$

And from 3 (*ii*) we get the following

$$\sum_{p=0}^{2\min(m,n)} l_p\, f_{\alpha}(m+n-p) = 0 \qquad n \neq m.$$

$$= \delta_n \qquad n = m.$$

4. Recurrence Relations. (i) We have

$$f_{\alpha+1}(n) = \int_0^{\infty} e^{-x}x^{\alpha+1}L_n^{\alpha+1}(x)F(x)\,dx.$$

$$= \int_0^{\infty} e^{-x}x^{\alpha}[(n+\alpha+1)L_n^{\alpha}(x)-(n+1)L_{n+1}^{\alpha}(x)]F(x)\,dx$$

so that $$f_{\alpha+1}(n) = (n+\alpha+1)f_{\alpha}(n)-(n+1)f_{\alpha}(n+1)$$

(ii) Again from, (Howell (1938) vol. 25)

$$n!\,L_n^{m-n}(x) = (-)^{n-m}m!\,L_m^{n-m}(x);$$

We obtain, $$n!\,f_{m-n}(n) = \int_0^\infty e^{-x}x^{m-n}n!\,L_m^{m-n}(x)F(x)dx.$$

$$= (-)^{n-m}m!\int_0^\infty e^{-x}x^{m-n}L_m^{m-n}(x)F(x,dx$$

$$= (-)^{n-m}m!\sum_{k=0}^{m}(k!)^{-1}(2n-2m)_k f_{m-n}(m-k).$$

5. Transform of the Derivative of F(x): Let $F(x)$ and $F'(x)$ be continuous and non-decreasing function of x.

Then $$\mathrm{T}\,\{F'(x)\} = \int_0^\infty e^{-x}x^{\alpha}L_n^{\alpha}(x)F'(x)dx$$

$$= \int_0^\infty e^{-x}x^{\alpha}L_n^{\alpha}(x)F(x)dx - \alpha\int_0^\infty e^{-x}x^{\alpha-1}L_n^{\alpha}(x)F(x)dx$$

$$-\int_0^\infty e^{-x}x^{\alpha}\left\{\frac{d}{dx}L_n^{\alpha}(x)\right\}\mathrm{F}(x)dx \quad \text{(on integration by parts)}$$

$$T\,\{F'(x)\} = f_\alpha(n) - \alpha\sum_{k=0}^{n} f_{\alpha-1}(k) + \sum_{k=0}^{-1} f_\alpha(k).$$

(Erdélyi (1953), p. 192)

$$T\,\{F''(x)\} = \int_0^\infty e^{-x}\,x^{\alpha}\,L_n^{\alpha}(x)\,F''(x)\,dx$$

$$= \int_0^\infty e^{-x}\,x^{\alpha}\,L_n^{\alpha}(x)\,F'(x)\,dx + \alpha\int_0^\infty \frac{d}{dx}\left[e^{-x}\,x^{\alpha}\,L_n^{\alpha}(x)\right]F(x)\,dx$$

$$-\alpha\int_0^\infty \frac{d}{dx}\left[e^{-x}\,x^{\alpha}\,L_{n-1}^{\alpha+1}(x)\right]F(x)\,dx$$

$$= T\,\{F'(x)\} - \alpha\int_0^\infty e^{-x}\,x^{\alpha-1}\,L_n^{\alpha}(x)\,F(x)\,dx$$

$$+\alpha(\alpha-1)\int_0^\infty e^{-x}\,x^{\alpha-2}\,L_n^{\alpha}(x)\,F(x)\,dx - 2\alpha\int_0^\infty e^{-x}\,x^{\alpha-1}\,L_{n-1}^{\alpha+1}(x)\,F(x)\,dx$$

$$+\int_0^\infty e^{-x}\,x^{\alpha}\,L_{n-1}^{\alpha+1}(x)\,F(x)\,dx + \int_0^\infty e^{-x}\,x^{\alpha}\,L_{n-2}^{\alpha+2}(x)\,F(x)\,dx,$$

$$T\,\{F'(x)\} - \alpha\sum_{m=0}^{n}(m!)^{-1}\,(1)_m\,f(n-m) + \alpha(\alpha-1)\sum_{m=0}^{n}(m!)^{-1}\,(2)_m\,f_{\alpha-2}(n-m)$$

$$-2\alpha\sum_{m=0}^{n-1}(m!)^{-1}\,(2)_m\,f_{\alpha-1}(n-m-1) + \sum_{m=0}^{n-1}(m!)^{-1}\,(1)_m\,f_\alpha\,(n-m-1)$$

$$+\sum_{m=0}^{n-2} (m!)^{-1}\,(2)_m\, f_\alpha\,(n-m-2) \quad \text{Utilising the relation (8)}$$

$$T\{F''(x)\} = f_\alpha(n) - 2\alpha \sum_{m=0}^{n} f_{\alpha-1}\,(n-m) + 2\sum_{m=0}^{n-1} f_\alpha\,(n-m-1)$$

$$-2\alpha \sum_{m=0}^{n-1} (m+1)\, f_{\alpha+1}\,(n-m-1) + \alpha(\alpha-1) \sum_{m=0}^{n} (m+1)\, f_{\alpha-2}\,(n-m)$$

$$+\sum_{m=0}^{n-2} (m+1)\, f^\alpha\,(n-m-2).$$

and so on for higher derivatives.

6. Inverse transformation. The inverse transform of $f_\alpha(n)$ is given by

$$F(x) = T^{-1}\,\{f_\alpha(n)\}. \qquad \text{defined in } (0,\ \infty)$$

Let

$$F(x) = \sum_{n=0}^{\infty} C_n\, L_n^\alpha(x).$$

Then by (4), we obtain,

$$F(x) = \sum_{n=0}^{\infty} (\delta_n)^{-1}\, f_\alpha(n)\, L_n^\alpha(x) = T^{-1}\,\{f_\alpha(n)\}$$

$$(0 < x < \infty)$$

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BENDING OF A SHALLOW SPHERICAL SHELL UNDER UNIFORM PRESSURES WITH THE BOUNDARY PARTLY CLAMPED AND PARTLY SIMPLY-SUPPORTED

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Summary. The transverse displacement in the bending of a spherical shell under uniform pressures on its surface has been found in accordance with the theory of shallow shells. The boundary is assumed to be partly clamped and partly simply supported The problem is formulated in terms of the Fredholm integral equation, the solution of which is obtained by elementary considerations A special problem is solved to illustrate the method.

1. Introduction. The general theory of thin elastic shells even with the assumption of small strain is too involved to solve special problems, Some simplification is obtained in the so called shallow shell theory in which the ratio of the maximum rise of the shell to its maximum span is much less than unity. General treatment of such a shallow shell was given by Marguerre (1938) while Federhofer (1937), Reissner (1946a, 1946b, 1955, 1956, 1959), Johnson and Reissner (1955, 1957) and others considered the bending, transverse vibrations and deformations of shallow shell of a spherical form. The present paper concerns with the bending of a shallow spherical shell under uniform pressures. The boundary conditions are assumed to be mixed; a part of the boundary is clamped while the remaining part is simply-supported. The problem is formulated in terms of an integral equation of the Fredholm type. The kernel function as well as the solution of the auxiliary problem required for the integral equation are obtained from the fundamental equations of the theory. Use of Dirac delta function puts the integral equation in such a form that the solution is obtained by elementary considerations. A numerical example has been worked out as an illustration and the effect of clamping a part of the boundary, otherwise simply-supported, has been shown in a tabular form.

2. Fundamental equations and boundary conditions. In the bending of a shallow spherical shell by transverse loads the differential equations to be solved was put by Reissner (1946b, 1959) in the forms

$$\nabla^4 F - \frac{Eh}{R}\nabla^2 w = 0 \qquad (2.1)$$

$$D\nabla^4 w + \frac{1}{R}\nabla^2 F = p(r,\theta) \qquad (2.2)$$

where F = Airy's stress function, E = modulus of elasticity, 'h' = wall thickness of the shell, R = radius of middle surface of the shell, w = transverse (axial) displacement,

$$D = Eh^3/[12(1-\nu^2)],$$

ν = Poisson ratio, p = transverse load and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

the Laplace operator in polar coordinates (r,θ). Stress resultants couples required in the present paper are

$$\left.\begin{aligned} N_r &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}, \quad N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) \\ M_r &= -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right). \end{aligned}\right\} \tag{2.3}$$

Let a be the radius of the base of the spherical shell. If a part of the boundary is clamped while the remaining part is simply-supported the boundary conditions on $r = a$ are

$$N_r = 0, \quad 0 \leqq \theta \leqq 2\pi \tag{2.4}$$

$$N_{r\theta} = 0, \quad 0 < \theta < 2\pi \tag{2.5}$$

$$w = 0, \quad 0 < \theta < 2\pi \tag{2.6}$$

$$\frac{\partial w}{\partial r} = 0, \quad 0 < \theta < \theta_1 \tag{2.7}$$

$$M_r = 0, \quad \theta_1 < \theta < 2\pi \tag{2.8}$$

There is the further condition that all functions are regular at $r = 0$. From (2.1) and (2.2),

$$\nabla^2(\nabla^4 + \lambda^4)\, w = \frac{1}{D}\nabla^2 p \tag{2.9}$$

$$\nabla^2(\nabla^4 + \lambda^4)\, F = \lambda^4 R p \tag{2.10}$$

with

$$\lambda^4 = Eh/(R^2 D) \tag{2.11}$$

3. Formulation of the integral equation. Let $\bar{w}(r, \theta; a, \varphi)$ denote the deflection of the shell at (r, θ) due to a concentrated edgemoment of unit strength acting at the point $r = a$, $\theta = \varphi$ when the whole of the boundary is simply-supported and no transverse load is acting. Then the deflection due to the moment of magnitude $M(a, \varphi)$ distributed on the part $r = a$, $0 < \varphi < \theta_1$, under the above boundary condition is

$$\int_0^{\theta_1} M(a, \varphi)\, \overline{w}(r, \theta; a, \varphi)\, d\varphi.$$

Let $\overline{\overline{w}}(r, \theta)$ denote the deflection at (r, θ) due to the load distribution $p(r, \theta)$ when the whole boundary is simply-supported. The total deflection $w(r, \theta)$ corresponding to the boundary conditions (2 4), (2.5), (2 6) and (2.8) is then

$$w(r, \theta) = \overline{\overline{w}}(r, \theta) + \int_0^{\theta_1} M(a, \varphi)\, \overline{w}(r, \theta; a, \varphi)\, d\varphi. \tag{3.1}$$

The unknown moment $M(a, \varphi$ is determined from the condition (2.7). Thus

$$\frac{\partial \overline{\overline{w}}(r, \theta)}{\partial r} + \int_0^{\theta_1} M(a, \varphi)\, \frac{\partial \overline{w}(r, \theta; a, \varphi)}{\partial r}\, d\varphi = 0, \; r = a, \; 0 < \theta < \theta_1. \tag{3.2}$$

Equation (3.2) is the Fredholm's integral equation of first kind and is solved in the usual way. The kernel function $\overline{w}$ as well as $\overline{\overline{w}}$, which is called the solution for the auxiliary problem can be found independently and then the equation (3.2) determine $M(a, \varphi)$. Finally w is obtained from (3 1). Similar formulation was given by Nowacki (1951, 1953, 1954) and Kaliski and Nowacki (1956) for rectangular plates and by Olesiak (1957) for cylindrical shells.

4. Simply-supported shell under a concentrated edge-moment. We assume that

$$p = 0; \; w = \sum_{n=0,1,2,\ldots}^{\infty} w_n(r) \cos n\theta, \qquad F = \sum_{n=0,1,2,\ldots}^{\infty} F_n(r) \cos n\theta.$$

These values satisfy (2.2), (2.9) and (2 10) if

$$\nabla_n^4 w_n + \frac{1}{RD} \nabla_n^2 F_n = 0 \tag{4.2}$$

$$\nabla_n^2 (\nabla_n^4 + \lambda^4)\, w_n = 0 \tag{4.3}$$

$$\nabla_n^2 (\nabla_n^4 + \lambda^2)\, F_n = 0 \tag{4 4}$$

where

$$\nabla_n^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}. \tag{4.5}$$

The solution of (4.3) is obtained by adding the solutions of $\nabla_n^2 w_n = 0, (\nabla_n^2 + i\lambda^2) w_n = 0$ and $(\nabla_n^2 - i\lambda^2) w_n = 0$. If the regularity condition at $r = 0$ is used we have

$$w_n = C_n r^n + \bar{A}_n J_n(\lambda r i^{\frac{1}{2}}) + \bar{B}_n I_n(\lambda r i^{\frac{1}{2}}) \tag{4.6}$$

where J_n and I_n denote Bessel function and modified Bessel function of order n respectively. Introducing the Kelvin function defined as (Johnson ann Reissner, 1957; Watson, 1948)

$$\left.\begin{aligned} J_n(x\, i^{3/2}) &= \operatorname{ber}_n x + i \operatorname{bei}_n x \\ I_n(x\, i^{3/2}) &= (\operatorname{ber}_n x - i \operatorname{bei}_n x) \exp(-\tfrac{1}{2} n\pi i) \end{aligned}\right\} \tag{4 7}$$

and using the results $J_n(ix) = i^n I_n(x)$, $J_n(x) = J_n(-x)$ and $I_n(x) = I_n(-x)$ we get

$$\left.\begin{aligned} J_n(\lambda r i^{\frac{3}{2}}) &= \mathrm{ber}_n\, \lambda r - i\, \mathrm{bei}_n\, \lambda r \\ I_n(\lambda r i^{\frac{1}{2}}) &= i^{-n}(\mathrm{ber}_n\, \lambda r + i\, \mathrm{bei}_n\, \lambda r, \end{aligned}\right\} \tag{4.8}$$

and hence (4.6) reduces to

$$w_n = C_n r^n + A_n\, \mathrm{ber}_n\, \lambda r + B_n\, \mathrm{bei}_n\, \lambda r \tag{4.9}$$

where $$A_n = \bar{A}_n + i^{-n}\bar{B}_n,\ B_n = (-\bar{A}_n + i^{-n} B_n)i.$$

Similarly the solution of (4.4) is

$$F_n = \gamma_n r^n + \alpha_n\, \mathrm{ber}_n\, \lambda r + \beta_n\, \mathrm{bei}_n\, \lambda r. \tag{4.10}$$

Since $$\nabla_n^2 (A_n\, \mathrm{ber}_n\, \lambda r + B_n\, \mathrm{bei}_n\, \lambda r) = \lambda^2 (B_n\, \mathrm{ber}_n\, \lambda r - A_n\, \mathrm{bei}_n\, \lambda r) \tag{4.11}$$

substitution from (4.9) and (4.10) in (4.2) gives

$$\nabla_n^4 \left[\lambda^2 (B_n\, \mathrm{ber}_n\, \lambda r - A_n\, \mathrm{bei}_n\, \lambda r) + \frac{1}{RD}(\alpha_n\, \mathrm{ber}_n\, \lambda r + \beta_n\, \mathrm{bei}_n\, \lambda r)\right] = 0.$$

This being true for all values of r, the coefficients of $\mathrm{ber}_n\, \lambda r$ and $\mathrm{bei}_n\, \lambda r$ must separately vanish. Hence

$$\alpha_n/(RD) = -\lambda^2 B_n, \qquad \beta_n/(RD) = \lambda^2 A_n \tag{4.12}$$

so that (4.10) reduces to

$$F_n = \gamma_n r^n - \lambda^2 RD\, (B_n\, \mathrm{ber}_n\, \lambda r - A_n\, \mathrm{bei}_n\, \lambda r). \tag{4.13}$$

The conditions from (2.4) to (2.6) are satisfied if

$$r\frac{\partial F_n}{\partial r} - n^2 F_n = 0, \quad r\frac{\partial F}{\partial r} - F_n = 0, \quad w_n = 0 \qquad \text{at } r = a.$$

By (4.9) and (4.13) these give

$$\begin{aligned} n(1-n)\gamma_n a^n - \lambda^2 RDA_n\, (n^2\, \mathrm{bei}_n\, \lambda a - \lambda a\, \mathrm{bei}_n'\, \lambda a) \\ + \lambda^2 RDB_n\, (n^2\, \mathrm{ber}_n\, \lambda a - \lambda a\, \mathrm{ber}_n' \lambda a) = 0 \end{aligned} \tag{4.14}$$

$$\begin{aligned} (n-1)\, \gamma_n a^n - \lambda^2 RDA_n\, (\mathrm{bei}_n\, \lambda a - \lambda a\, \mathrm{bei}_n' \lambda a) \\ + \lambda^2 RDB_n\, (\mathrm{ber}_n\, \lambda a - \lambda a\, \mathrm{ber}_n' \lambda a) = 0 \end{aligned} \tag{4.15}$$

$$C_n a^n + A_n\, \mathrm{ber}_n\, \lambda a + B_n\, \mathrm{bei}_n\, \lambda a = 0 \tag{4.16}$$

the prime denoting differentiation with respect to the argument

Let it be assumed that

$$M_r(a, \theta) = \sum_{n=0,1,2,\ldots}^{\infty} M_n \cos_n \theta. \tag{4.17}$$

Substituting the values of w from the second equation of (4.1) in the third equation of (2.3), putting $r = a$ and then comparing it with (4.17) we get

$$\left[\frac{\partial^2}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)\right] w_n = -\frac{M_n}{D}, \quad r = a \qquad (4.18)$$

which with the help of (4.9) and (4.11) gives

$$n(1-n)C_n a^n + A_n(\lambda a \operatorname{ber}_n'\lambda a - n^2 \operatorname{ber}_n \lambda a + \frac{\lambda^2 a^2}{1-\nu} \operatorname{bei}_n \lambda a)$$

$$+B_n(\lambda a \operatorname{bei}_n'\lambda a - \frac{\lambda^2 a^2}{1-\nu} \operatorname{ber}_n \lambda a - n^2 \operatorname{bei}_n \lambda a) = \frac{M_n a^2}{D(1-\nu)} \qquad (4.19)$$

Equations (4.14), (4.15), (4.16) and (4.19) determine the coefficients as

$$\left.\begin{aligned} A_n &= -(M_n a^2 X_n)/[D(1-\nu)\Delta_n],\ B_n = -(M_n a^2 Y_n)/[D(1-\nu)\Delta_n] \\ C_n &= [M_n a^{2-n}(X_n \operatorname{ber}_n \lambda a + Y_n \operatorname{bei}_n \lambda a)]/[D(1-\nu)\Delta_n], \\ \gamma_n &= -(\lambda^3 R M_n Z_n)/[(1-\nu)a^{n-3}\Delta_n] \end{aligned}\right\} \qquad (4.20)$$

where

$$X_n = n \operatorname{ber}_n \lambda a - \lambda a \operatorname{ber}_n' \lambda a \qquad (4.21)$$

$$Y_n = n \operatorname{bei}_n \lambda a - \lambda a \operatorname{bei}_n' \lambda a \qquad (4.22)$$

$$Z_n = \operatorname{ber}_n' \lambda a \operatorname{bei}_n \lambda a - \operatorname{bei}_n' \lambda a \operatorname{ber}_n \lambda a \qquad (4.23)$$

$$\Delta_n = X_n^2 + Y_n^2. \qquad (4.24)$$

If a concentrated moment of unit strength acts at $\theta = 0$, we have $M_r(a, \theta) = \delta(\theta)$, where $\delta(\theta)$ is the Dirac delta function defined as

$$\delta(\theta) = 0,\ \theta \neq 0;\ \int_{-\infty}^{\infty} \delta(\theta)\, d\theta = 1;\ \int_{-\infty} f(\theta)\, \delta(\theta-\theta_0) d\theta = f(\theta_0). \qquad (4.25)$$

Then from (4.17)

$$M_n = \frac{1}{\pi}\int_{-\pi}^{\pi} M_r(a, \theta) \cos n\theta\, d\theta = \frac{1}{\pi}\int_{-\infty}^{\infty} \delta(\theta) \cos n\theta\, d\theta = \frac{1}{\pi} \qquad (4.26)$$

so that

$$\delta(\theta) = \frac{1}{\pi} \sum_{n=0,1,2,\ldots}^{\infty} \cos n\theta. \qquad (4.27)$$

Finally from (4.1), (4.9), (4.20) and (4.26) we get

$$w(r, \theta;\ a, 0) = -\frac{a^2}{\pi(1-\nu)D} \sum_{n=0,1,2,\ldots} \Big[\{X_n \operatorname{ber}_n \lambda r + Y_n \operatorname{bei}_n \lambda r$$

$$-(r/a)^n(X_n \operatorname{ber}_n \lambda a + \lambda_n \operatorname{bei}_n \lambda a)\} \times \frac{\cos n\theta}{\Delta_n}\Big]$$

The displacement $\bar{w}(r, \theta;\, a, \varphi)$ at (r, θ) due to a unit concentrated moment at $r = a$, $\theta = \varphi$ is obtained from the above expression by changing the coordinate θ to $\theta - \varphi$. Thus

$$\bar{w}(r, \theta;\, a, \varphi) = -\frac{a^2}{\pi(1-\nu)D} \sum_{n=0,1,2,}^{\infty} \Big[\{ X_n \operatorname{ber}_n \lambda r + Y_n \operatorname{bei}_n \lambda r - (r/a)^n (X_n \operatorname{ber}_n \lambda a + Y_n \operatorname{bei}_n \lambda a)\} \times \frac{\cos n(\theta-\varphi)}{\Delta_n}\Big]. \tag{4.28}$$

5. Simply-supported shell under uniform pressures If the shell is bent by uniform pressures $p = p_0$ (a constant) and the whole of the boundary be simply-supported all functions will be independent of (θ) and the equations (2.1) and (2.2) reduce to

$$\nabla_0^4 F - \frac{Eh}{R} \Delta_0^2 w = 0, \quad D\, \nabla_0^4 w + \frac{1}{R} \nabla_0^2 F = p_0 \tag{5.1}$$

where $\nabla_0^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial}{\partial r}$ in conformity with (4.5) noting that in this case $n = 0$. A particular solution of (5.1) is $w = 0$ and $F = \frac{1}{4} R p_0 r^2$. The complementary functions corresponding to (5.1) are the solutions of (4.3) and (4.4) and are of the forms (4.9) and (4.13) with $n = 0$. Thus writing $\overset{=}{w}$ for w we have

$$\overset{=}{w} = s_1 + s_2 \operatorname{ber} \lambda r + s_3 \operatorname{bei} \lambda r, \quad F = \frac{1}{4} R p_0 r^2 + s_4 - \lambda^2 R D\, (s_3 \operatorname{ber} \lambda r - s_2 \operatorname{bei} \lambda r) \tag{5.2}$$

if the Kelvin function of order zero are written in Thomson's notation $\operatorname{ber}_0 x = \operatorname{ber} x$, $\operatorname{bei}_0 x = \operatorname{bei} x$. Since only derivatives of the stress function F are required we may take s_4 as zero. The condition (2.5) is automatically satisfied while (2.4) and (2.6) give

$$s_3 \operatorname{ber}' \lambda a - s_2 \operatorname{bei}' \lambda a = (a p_0)/(2\, \lambda^3 D) \tag{5.3}$$

$$s_1 + s_2 \operatorname{ber} \lambda a + s_3 \operatorname{bei} \lambda a = 0. \tag{5.4}$$

With the help of the third equation of (2.3) and the formulae (14)

$$\operatorname{ber}'' x = -\operatorname{bei} x - x^{-1} \operatorname{ber}' x, \quad \operatorname{bei}'' x = \operatorname{ber} x - x^{-1} \operatorname{bei}' x$$

the condition $M_r = 0$ on $r = a$ gives

$$s_2 [(1-\nu) \operatorname{ber}' \lambda a + \lambda a \operatorname{bei} \lambda a] + s_3 [(1-\nu) \operatorname{bei}' \lambda a - \lambda a \operatorname{ber} \lambda a] = 0. \tag{5.5}$$

Equations from (5.3) to (5.5) determine the coefficients as

$$s_1 = (a p_0 S_1)/(2 \lambda^3 D \Delta) \tag{5.6}$$

$$s_2 = -(a p_0 S_2)/(2 \lambda^3 D \Delta) \tag{5.7}$$

$$s_3 = (a p_0 S_3)/(2 \lambda^3 D \Delta) \tag{5.8}$$

where
$$\left.\begin{aligned}
S_1 &= (1-\nu)\ \mathrm{ber}\ \lambda a\ \mathrm{bei}'\ \lambda a-\mathrm{bei}\ \lambda a\ \mathrm{ber}'\ \lambda a)-\lambda a\ \{(\mathrm{ber}\ \lambda a)^2+(\mathrm{bei}\ \lambda a)^2\}\\
S_2 &= (1-\nu)\ \mathrm{bei}'\ \lambda a-\lambda a\ \mathrm{ber}\ \lambda a\\
S_3 &= (1-\nu)\ \mathrm{ber}'\ \lambda a+\lambda a\ \mathrm{bei}\ \lambda a\\
\Delta &= (1-\nu)\ \{(\mathrm{ber}'\ \lambda a)^2+(\mathrm{bei}'\ \lambda a)^2\}+\lambda a(\mathrm{bei}\ \lambda a\ \mathrm{ber}'\ \lambda a-\mathrm{ber}\ \lambda a\ \mathrm{bei}'\ \lambda a).
\end{aligned}\right\}\quad(5.9)$$

6. Solution of the integral equation. The term $\frac{\partial \overline{\overline{w}}}{\partial r}$ $(r = a,\ \ 0<\theta<\theta_1)$ occuring in (3.2) is obtained from the first equation of (5.2) by utilising (5.7) and (5.8). Thus

$$\left(\frac{\overline{\overline{w}}}{\partial r}\right)_{r=a} = (a^2p_0X)/(2\lambda D\Delta) \tag{6.1}$$

where
$$X = \mathrm{ber}\ \lambda a\ \mathrm{ber}'\ \lambda a+\mathrm{bei}\ \lambda a\ \mathrm{bei}'\ \lambda a \tag{6.2}$$

With the help of (4.21), (4.22) and (4.24) the relation (4.28) gives

$$\left(\frac{\partial \overline{w}}{\partial r}\right)_{r=a} = \frac{a}{\pi\ D(1-\nu)}\sum_{n=0,1,2,}^{\infty} \cos n(\theta-\varphi).$$

If the result (4.27) is used the above relation reduces to

$$\left(\frac{\partial \overline{w}}{\partial r}\right)_{r=a} = \frac{a}{D(1-\nu)}\ \delta(\theta-\varphi). \tag{6.3}$$

Substitution from (6.1) and (6.3) in (3.2) gives

$$\int_0^{\theta_1} M(a,\varphi)\ \delta(\theta-\varphi)\ d\varphi = -[ap_0X(1-\nu)]/(2\lambda\Delta),\quad 0<\theta<\theta_1.$$

Since $\delta(\theta-\varphi) = \delta(\varphi-\theta)$ and $\delta(\theta-\varphi) = 0,\ \ \varphi\neq\theta,\ \ (0<\theta<\theta_1)$ the last equation is written as

$$\int_{-\infty}^{\infty} M(a,\varphi)\ \delta(\varphi-\theta)\ d\varphi = -[ap_0X(1-\nu)]/(2\lambda\Delta),\quad 0<\theta<\theta_1$$

the solution of which is

$$M(a,\varphi) = -[ap_0X(1-\nu)]/(2\lambda\Delta),\quad 0<\varphi<\theta_1 \tag{6.4}$$

Substituting in (3.1) from (4.28), (6.4) and the first one of (5.2) we obtain

$$[(2\lambda^3D\ \Delta w)/(p_0a) = W_s+W_c \tag{6.5}$$

where
$$W_s = S_1-S_2\ \mathrm{ber}\ \lambda r+S_3\ \mathrm{bei}\ \lambda r \tag{6.6}$$

$$\begin{aligned}
[(\pi W_c)/(a^2\lambda^2X)] = {} & [\lambda a\{(\mathrm{ber}'\ \lambda a)^2+(\mathrm{bei}'\ \lambda a)^2\}]^{-1}\ \{X-(\mathrm{ber}'\ \lambda a\ \mathrm{ber}\ \lambda r+\mathrm{bei}'\ \lambda a\ \mathrm{bei}\ \lambda r)\}\theta_1\\
& +\sum_{n=0,1,2\ldots}^{\infty}[(n\Delta_n)^{-1}\ \{X_n\ \mathrm{ber}_n\ \lambda r+Y_n\ \mathrm{bei}_n\ \lambda r-(r/a)^n\ (X_n\ \mathrm{ber}_n\ \lambda a+Y_n\ \mathrm{bei}_n\ \lambda a)\}\\
& \times\{\sin n\theta-\sin n(\theta-\theta_1)\}].
\end{aligned} \tag{6.7}$$

Here $W_s/(2\Delta)$ is proportional to the displacement for the simply-supported boundary and $W_c/(2\Delta)$ is proportional to the additional displacement due to clamping a part of the boundary (both expressed in dimensionless forms). As a numerical example we take $\theta_1 = \pi/2$, $\lambda a = 2.0$, $\nu = 0.3$ and evaluate $W_s/(2\Delta)$, $W_c/(2\Delta)$ and $[(\lambda^3 Dw)/(p_0 a)] = (W_s + W_c)/(2\Delta)$ along the symmetrical radial line $\theta = \pi/4$. In (6.7) terms upto $n = 1$ are retained and the Kelvin functions of order one and their derivatives are calculated from the corresponding Bessel functions (Jahnke and Emde, 1945). The results are given in the table that follows. It is seen that the maximum effect due to clamping a part of the boundary occurs near about the point $(r/a) = 0.4$, $\theta = \pi/4$.

TABLE

r/a	0.0	0.2	0.4	0.6	0.8	1.0
$W_s/(2\Delta)$	1.6370	1.5594	1.3317	0.9710	0.5094	0.0
$-W_c/(2\Delta)$	0.5946	0.6801	0.6951	0.6091	0.3887	0.0
$[\lambda^3 Dw)/(p_0 a)] = (W_s + W_c)/(2\Delta)$	1.0424	0.8793	0.6366	0.3619	0.1207	0.0

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ABSTRACT OF PROCEEDINGS OF THE SYMPOSIUM ON ELASTICITY, PLASTICITY AND RHEOLOGY UNDER THE AUSPICES OF THE CALCUTTA MATHEMATICAL SOCIETY HELD IN OCTOBER, 1959

PRESIDENT—PROF. DR. S. N. BOSE, F. R. S., *(National Professor)*

Inaugural address—Prof. B. Sen, D.Sc., F.N.I.

SPEAKERS

Bibhutibhusan Sen—Direct methods of solving two-dimensional problems of elasticity.

S. C. Das Gupta—Love waves in stratified layers.

Saktikanta Chakraborty—Disturbances in elastic media.

M. Mitra—The source problem is seismology.

B. Karunes—Shock wave propagation effects in impact problems of plasticity.

Avtar Singh—Thermo-elastic problems for anisotropic media.

P. P. Chattarji—Finite strain theory.

Gunadhar Paria—Porous elastic media.

P. D. S. Verma—Hypo-elasticity.

D. N. Mitra—Application of function-theoretic method to problems of elasticity.

S. Dutt—Effects of cross elasticity.

DIRECT METHODS OF SOLVING TWO-DIMENSIONAL PROBLEMS OF ELASTICITY

By

BIBHUTIBHUSAN SEN, *Jadavpur, Calcutta.*

General methods of solving two-dimensional boundary-value problems of elasticity by the systematic use of the complex variable theory are now well known (Muskhelishvili, 1958). It is, however, found that in many cases direct methods make the solution simpler.

To demonstrate the utility of direct methods, we shall begin with the plane problem of non-isotropic material in which the stress strain relations are

$$e_{xx} = E_1^{-1}\widehat{xx} - k\widehat{xy}, \quad e_{yy} = -k\widehat{xx} + E_2^{-1}\widehat{yy}, \quad e_{xy} = \mu^{-1}\widehat{xy},$$

such that

$$k = \sigma_1/E_2 = \sigma_2/E_1.$$

From the equations of equilibrium and the compatibility relations, it is possible to deduce expressions for stress components and components of displacement in terms of two complex functions of z_1 and z_2 where $z_1 = x + i\gamma_1 y$, $z_2 = x + i\gamma_2 y$, γ_1, γ_2 being real quantities depending on elastic constants.

As a simple example, let us suppose that on the straight boundary $y = 0$, of a semi-infinite plate, $\widehat{yy}$ is prescribed, while $\widehat{xy} = 0$. Then we can put

$$\widehat{xy} = \mathbf{Re}[i\gamma_1\gamma_2\{\varphi(z_2) - \varphi(z_1)\}]],$$

$$\widehat{yy} = \mathbf{Re}[\gamma_2\varphi(z_1) - \gamma_1\varphi(z_2)],$$

$$\widehat{xx} = \mathbf{Re}[\gamma_1\gamma_2\{\gamma_2\varphi(z_2) - \gamma_1\varphi(z_1)\}].$$

It is evident that

$$(\widehat{xy})_{y=0} = 0 \text{ and } (\widehat{yy})_{y=0} = (\gamma_2 - \gamma_1)[\mathbf{Re}\varphi(z)]_{y=0}$$

where $z = x + iy$. $(\widehat{yy})_{y=0}$ being prescribed, the form of the φ function can easily be determined (Sen, 1954).

Next we shall consider the first boundary value problem of an elastic plate of isotropic material with a circular boundary. If $\widehat{rx}$ and $\widehat{ry}$ be the stress components in the x and y directions on a circle of radius a, we can, in the absence of body forces, deduce the equations

$$r.\widehat{rx} = \mathbf{Re}\left[\frac{r^2-a^2}{4}\left\{\frac{f(z)-zf'(z)}{z}\right\}\right]+aL(z),$$

$$r.\widehat{ry} = \mathbf{Re}\left[\frac{r^2-a^2}{4}\,i\,\left\{\frac{f(z)-zf'(z)}{z}\right\}\right]+aM(z),$$

where $L(z)$ and $M(z)$ are analytic functions of z, such that

$$[\widehat{rx}]_{r=a} = [\mathbf{Re}\ L(z)]_{r=a} \text{ and } [\widehat{ry}]_{r=a} = [\mathbf{Re}\ M(z)]_{r=a}.$$

From the relation $\widehat{xx}+\widehat{yy} = \mathbf{Re}\, f(z)$ we get in general

$$f(z) = 2a\,\frac{L(z)+iM(z)}{z}, \quad \text{(Sen, 1946).}$$

Thus when $[\widehat{rx}]_{r=a}$ and $[\widehat{ry}]_{r=a}$ are known, we can find the functions $L(z)$, $M(z)$ and hence $f(z)$.

For the solution of the corresponding second boundary problem, we can deduce the expressions for the displacements u and v in the directions of x and y in the forms

$$u = \frac{1}{4(1-2\sigma)}\ \mathbf{Re}\left[(a^2-r^2)\,\frac{f(z)}{z}\right]+\mathbf{Re}\ g_1(z),$$

$$v = \frac{1}{4(1-2\sigma)}\ \mathbf{Re}\left[(a^2-r^2)\frac{if(z)}{z}\right]+\mathbf{Re}\ g_2(z),$$

such that

$$[\mathbf{Re}\ g_1(z)]_{r=a} = [u]_{r=a} \text{ and } [\mathbf{Re}\ g_2(z)]_{r=a} = [v]_{r=a}.$$

For generalized plane stresses σ is modified Poisson's ratio.

From the relation

$$\mathbf{Re}\ f(z) = \triangle = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}, \text{ we get}$$

$$f(z) = \frac{2(1-2\sigma)}{3-4\sigma}\ [g'_1(z)+ig'_2(z)]. \text{ [Sen, 1957]}$$

Since $g_1(z)$ and $g_2(z)$ are supposed to be known, the value of $f(z)$ and hence those of u and v can be found.

In the case of curvilinear boundaries other than the circle, results cannot be put in as convenient terms as in the previous cases. But a direct tentative method is sometimes found useful. If $z = f(\zeta)$ where $\zeta = \xi+i\eta$ and the bounding curve is given by ξ = constant or η = constant, we find that the stress components $\widehat{\xi\xi}$, $\widehat{\eta\eta}$ and $\widehat{\xi\eta}$ satisfy the relations

$$8J^2\widehat{\xi\xi} = \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} + 4J^2\odot + F,$$

$$8J^2\widehat{\eta\eta} = \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} + 4J^2\odot - F,$$

$$8J^2\widehat{\xi\eta} = -\frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \xi}\frac{\partial \odot}{\partial \eta} - G,$$

where

$r^2 = x^2+y^2$, $\odot = \widehat{\xi\xi}+\widehat{\eta\eta}$, $J = \left|\frac{dz}{d\zeta}\right|$, and $F+iG$ is an analytic function of z, (Sen, 1948).

By assuming tentative values of $\odot$, expressions for F and G can be found in a simple manner in many cases from the known values of the stresses on the boundary.

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LOVE WAVES IN STRATIFIED LAYERS

By

S. C. Das Gupta, *Calcutta*

The existence of long waves with large horizontal transverse component of displacement in actual seismograms led A.E.H. Love to the discovery of the possibility of waves with such characteristics when there is a layered structure. Alternative condition which may give rise to the waves of the same characteristics was found by E. Meissner (1921) and also independently by K. Sezawa (1926) and the lack of Meissner's theory was supplemented by K. Aichi (1922).

The effect of double superficial layer on the velocity of propagation of Love waves was examined independently by R. Stoneley (1925) and Matuzawa (1927) and the effect of heterogeneity in the lower layer was also examined independently by H. Jeffreys (1928) and T. Metuzawa (1929) by taking a homogeneous silicic layer over a mantle with a gradient of shear velocity instead of considering two layers over a homogeneous medium. Das Gupta (1951 & 52) considered similar layers with the lower medium varying in density and rigidity under different simplified laws such as linear, exponential or square of depth. However, these are simplification of the problem. A special character of Love waves is its dispersion. Love waves with periods less that 20 secs, showing inverse dispersion according to the theoretical curve have not been observed. The discrepancies for the periods less than 20 secs may be due to a gradient in crust or to the increased effects of crustal heterogeneity on the shorter period waves or both For periods greater than 20 secs, Love wave dispersion is found to agree with the results of seismic refractions in finding a non-layered, silicic continental crust having an average thickness of 35 Kms.

Recently K. Tazmine and H. Okada (1958) have undertaken theoretical considerations of Love waves in three layers to represent interesting characters. The frequency equation is given by

$$\tan \eta_1 H_1 = \left(i\frac{\mu_2\eta_2}{\mu_1\eta_1}\right)\cdot\frac{i\left\{\tan\eta_2 H_2-\left(\frac{i\mu_3\eta_3}{\mu_2\eta_2}\right)\right\}}{1+\left(\frac{i\mu_3\eta_3}{\mu_2\eta_2}\right)\tan\eta_2 H_2}$$

where $\xi^2+\eta_j^2 = k_j^2 (j = 1, 2, 3),\ k_j = \omega/v_j,\ v_j = (\mu_j/\zeta_j)^{\frac{1}{2}}$

and wave is of the form $e^{i(\omega t-\xi x)}(A_j e^{i\eta_j z}+B_j e^{-i\eta_j z})$

Group velocities, obtained from phase velocities, H_2/H_1 being taken as parameter shows two minima when a particular value is assumed for μ_3/μ_2. Corresponding displacements will have two maxima. However, observations are to be made in the line.

The problem of generation of Love-waves were treated by K. Sezawa (1935) adopting the procedure shown by Nakano (1925). Sato (1952) confined his attention to retardation of phases in Love waves. Due to a cylindrical source which is situated at (0, E) in superficial layer expressed by

$$v_0 = H_0{}^2\,(kr)\exp(ipt),$$

displacement v is given by

$$v = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{2}{\beta}\left\{\exp\left[-\beta(H-E)\right]+\frac{q}{C(f)}\exp\left[-\beta(H+E)\right]+\frac{q}{C(f)}\exp\left[-\beta(3H-E)\right]\right\}$$

$$\times\exp(ifx)\,df\ldots(a)$$

$$C(f) = 1-q\exp(-2\beta H),\; q=\frac{\beta-\chi\beta'}{\beta+\chi\beta'},\; \chi=\frac{\mu'}{\mu},\quad H = \text{height of layer}$$

From the second term of (a), the travel time is determined. If we deform the path $\int_{-\infty}^{\infty}$ into that which passes the saddle point, then the real coordinates of points where the path of steepest descent crosses the axis will be $-kr/x$ and $-kx/r$. Here $r=(H+E)^2+x^2$. When the epicentral distance becomes large, $-kx/r=-K$ where $-K$ is the solution of the characteristic equation $C(f)=0$. From this,

$$x = x'_0 = (H+E)\Bigg/\sqrt{\frac{V^2_{\text{Love}}}{V_s^{2}}} \qquad (b)$$

Therefore, the critical distance x_0' where the Love waves can be observed may be expressed by the formula (b). The previous paragraph and this one may help us in determining the structure of the crust.

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DISTURBANCE IN ELASTIC MEDIA

By

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Jeffreys (1931) considered a disturbance in an infinite elastic medium as generated by a suddenly applied pressure $AH(t)$ on the inner surface of a spherical cavity $r = a$ situated inside the medium. The displacement at a point P distant r from the centre of the cavity is purely radial and is given by (assuming Poisson's relation)

$$u_r = 0, \quad \text{for} \quad t' < 0,$$

$$= \frac{Aa^3}{4\mu r^2}\left[1+\left\{\left(\frac{r}{a}-\tfrac{1}{2}\right)\sqrt{2}\sin\left(\frac{2\sqrt{2}\alpha t'}{3a}\right)-\cos\left(\frac{2\sqrt{2}\alpha t'}{3a}\right)\right\}\exp\left\{-\frac{2\alpha t'}{3a}\right\}\right], \tag{1}$$

$$\text{for} \quad t' > 0,$$

$$\equiv \Psi(t'), \text{ say,}$$

where $$t' = t - (r-a)/\alpha$$

At large distances, it is approximately

$$u_r = \frac{\sqrt{2}}{4\mu r}\sin\left(\frac{2\sqrt{2}\alpha t'}{3a}\right)\exp\left(-\frac{2\alpha t'}{3a}\right) \tag{2}$$

for not very large time t.

Equation (2) shows that the resulting disturbance propagates with velocity α in all directions from the centre of the cavity and the displacement at any point is zero initially, rises to a maximum and then decays exponentially to a constant value. Das Gupta (1954) considered the disturbance as generated by an impulsive pressure $A\delta(t)$ where $\delta(t)$ is the impulse function called Dirac-delta function. The resulting displacement is

$$u_r = 0 \quad \text{for} \quad t' < 0$$

$$= \frac{Aa^2\alpha}{2\sqrt{2}\mu r^2}\left[\left(1-\frac{2r}{3a}\right)\sin\frac{2\sqrt{2}\alpha}{3a}t' + \frac{2\sqrt{2}r}{3a}\cos\frac{2\sqrt{2}\alpha}{3a}t'\right]\exp\left(-\frac{2\alpha t'}{3a}\right), \quad \text{for} \quad t' > 0. \tag{3}$$

In this case the displacement at the time of arrival of the pulse is non-zero, and it increases afterwards to a maximum before dwindling down to zero.

If however a constant pressure acts for a finite time 't_0', on the spherical cavity, the corresponding displacement is

$$\begin{aligned} u_r &= 0, \quad \text{for} \quad t' < 0 \\ &= \Psi(t'), \quad \text{for} \quad 0 < t' < t_0 \\ &= \Psi(t') - \Psi(t'-t_0), \quad \text{for} \quad t' > t_0. \end{aligned}$$

In this case, the displacement at any point is the same as in Jeffrey's case for time t_0 after which it diminishes exponentially to zero.

Recently A.C. Eringen (1957) considered an elastodynamic problem concerning a spherical cavity. He has applied the method of Fourier transform to solve the general problem of disturbances generated by applied forces on the inner surface of the cavity.

It was shown by Jeffreys (1943) that the analogy of a cylindrical pulse can explain the diffuseness of the seismological pulses. Problems of disturbances of cylindrical origin have been considered by Sezawa (1927) in the case of periodic pressure ; Kromm (1948), in the case of suddenly applied pressure, Chakrabarty (1957), in the case of an impulsive pressure, and in the case of periodic forces applied over a band of the surface of a cylindrical hole (1956).

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THE SOURCE PROBLEM IN SEISMOLOGY.

By

M. Mitra, *Jadavpur, Calcutta*

An important problem in seismology is that of furnishing an explanation of the rapid oscillations which are observed on a seismograph, recording the ground motion near it, during the initial stages of the motion caused by a distant earthquake.

Jeffreys (1931) calculated the displacement produced at distant points in an infinite, isotropic, elastic medium due to the application of impulsive strseses on the surface of a spherical cavity in the medium. His results showed oscillations in the expression for the displacement but the amplitude decreased very rapidly.

Other authors used a different model for a source; they considered it to be situated along a line or at a point placed at a finite distance from a free, plane boundary representing the earth's surface. It is known that, in the absence of body forces, the equations of motion of an isotropic, elastic medium are satisfied if the displacement $\vec{q}$ is given by

$$\vec{q} = \operatorname{grad} \varphi + \operatorname{rot} \vec{A}$$

where φ and $\vec{A}$ satisfy the wave equations

$$\nabla^2 \varphi - \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} = 0$$

$$\nabla^2 \vec{A} - \frac{1}{\beta^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \qquad (ii)$$

α, β being the compressional and shear wave velocities in the medium. The source is then represented by functions φ_0 or $\vec{A}_0$, according as the source is compressional or distortional in nature, satisfying equations (*i*) or (*ii*) everywhere in the semi-infinite medium except on a sphere which for $t = 0$ reduces to a point, called the source.

The following expressions have been used by various authors:

(a) $\varphi_0 = \frac{a}{2} H(t - r/\alpha) \cosh^{-1} \alpha t/r$, ... Lapwood (1949), Garvin (1956)

(b) $\varphi_0 = \frac{1}{R} H(t - R/\alpha)$, ... Pinney (1954)

(c) $\vec{A}_0 = \frac{\vec{K}}{R} H(t - R/\beta)$, ... Pinney (1954)

where (r, θ) are polar co-ordinates, (R, θ, ψ) spherical polar co-ordinates and $\vec{K}$

a constant vector, the pole being in each case taken at the source.

The solution of the problem is usually obtained by expressing $\vec{q}$ in the form

$$\vec{q} = \text{grad}\,(\varphi + \varphi_0) + \text{rot}\,(\vec{A} + \vec{A}_0)$$

so as to satisfy the boundary conditions by the help of φ, $\vec{A}$ which satisfy equations (*i*), (*ii*) respectively everywhere in the medium. Thus, if body forces are assumed to act, it is seen that the equation of motion are satisfied provided the body force $\vec{F}$ is given by

$$\vec{F} = \text{grad}\left[\frac{\partial^2\varphi_0}{\partial t^4} - \alpha^2\nabla^2\varphi_0\right] + \text{rot}\left[\frac{\partial^2\vec{A}_0}{\partial t^2} - \beta^2\nabla^2\vec{A}_0\right]$$

$$= \text{grad}\,P_0 + \text{rot}\,\vec{Q}_0 = 0, \quad \text{everywhere in the medium except at the source,}$$

where

$$\frac{\partial^2\varphi_0}{\partial t^2} - \alpha^2\nabla^2\varphi_0 = P_0,$$

$$\frac{\partial^2\vec{A}_0}{\partial t^2} - \beta^2\nabla^2\vec{A}_0 = \vec{Q}_0.$$

This result permits of an interpretation of the source in terms of the body force acting there. Thus it can be shown that in cases (a) and (c), $\vec{F}$ is given by $H(t)$ grad $[\delta(x)\delta(y)]$ and $H(t)$ grad $[\delta(x)\delta(y)\delta(z)]$ respectively. Again by taking

$$P_0 = H(t)\,\delta(y)\,\delta(x - Vt),\; V < \beta,$$

it is possible to consider a moving source.

Recently, Pekeris and Lifson (1957) solved the source problem for a concentrated point force. Also the author (Mitra) has determined the surface displacement in the two-dimensional problem for simple types of body forces:

(d) $X = 0,\; Y = \delta(x)\delta(y)H(t),$
(e) $X = 0,\; Y = f(x,y)H(t),$
(f) $\vec{F} = H(t)\,\text{grad}\,f(x,y),$

where $f(x,y) = 1$, inside the rectangle bounded by $|x| = a$, $|y - h| = b$, $f(x,y) = 0$, elsewhere, and the functions are taken in the distributional sense.

However, none of the numerical calculations carried out so far by different authors show rapid oscillations of the displacement in the initial stages; the mathematical models hitherto used have, therefore, proved inadequate in this respect.

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SHOCK WAVE PROPAGATION EFFECTS IN IMPACT PROBLEMS OF PLASTICITY

By

B. Karunes, *Calcutta*

In the problem of impulsive loading the initial conditions involve discontinuities of velocity at certain boundaries. Rahmatulin (1945) has shown that such discontinuous initial conditions enable us to construct the solution to the problem with the help of waves along which discontinuities in slope or stress propagate. He had dealt with the case of direct impact on elastic strands (1947). Craggs (1954) has given a solution for the case of transverse impact on plastic-elastic strings. Karunes and Onat (1958) were interested in similar problems related to plates and membranes where the discontinuous change in the cross-sectional area at the sections of stress discontinuity was important. Here the equations governing such waves of discontinuity, called shock waves borrowing the name from similar discontinuous wave fronts in gas dynamics are briefly reviewed. A comprehensive bibliography on the subject is given in a recent book by Cristescu (1958).

Force-extension relation. Neglecting elastic strains and considering the material as rigid upto the yield point and assuming the material to be strain hardening having a piece-wise linear force-extension, we have

$$f = f_0 + \lambda |\varepsilon| \tag{1}$$

where f denotes the membrane force at any instant measured on its cross-section normal to the plane of motion, ε is the longitudinal strain defined as the change in length per unit initial length in the plane of motion, λ is the slope of strain-hardening line and f_0 is the force required on the membrane for its yielding in its original cross-section.

If a_0 and a be the initial and current cross-sectional areas, then since the material is incompressible

$$\varepsilon = \frac{a_0 - a}{a} \tag{2}$$

so that

$$f = f_0 + \lambda \left| \frac{a_0 - a}{a} \right| \tag{1a}$$

Shock-Wave Relations. Let the longitudinal shock-wave move with a volocity V while the material at its front and back move with velocities U_1 and U_2 and let the corresponding areas be a_1 and a_2 and the corresponding forces be f_1 and f_2.

Conservation of mass during any time interval t gives

$$a_1(U_1 - V) = a_2(U_2 - V) \tag{3}$$

Conservation of momentum for the same time interval gives

$$f_1 - f_2 = \varrho a_1 (U_1 - V)(U_1 - U_2) \tag{4}$$

where ϱ is the density of the material.

Transverse wave Relations. The transverse wave produces a discontinuity in slope. Upto the point of discontinuity, it moves with a velocity u and the motion of the other part consists of a rigid body translation characterised by the velocity components U and V. The law of conservation of mass for a time interval t gives

$$\left.\begin{aligned} U &= u(1-\cos\theta) \\ V &= \quad -u\sin\theta \end{aligned}\right\} \tag{5}$$

where θ is the slope of discontinuity.

The law of conservation of momentum gives

$$\left.\begin{aligned} f_1 &= \varrho a_1 u^2 \\ f_2 &= \varrho a_1 u^2 \end{aligned}\right\} \tag{6}$$

Particular problems of (1) Direct impact on a rod of uniform section and (2) Transverse impulse on a uniform bar are discussed in detail showing that the variation of cross-section and stress can be fully calculated by the use of above equations. In each of the examples it is seen that impact velocity determines the transverse wave velocity whereas the shock wave velocity is dependent only on the yield stress and the strain hardening rate. Thus the transverese wave velocity may be less than equal to or greater than the shock wave velocity and on this depends the mode of deformation.

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THERMO-ELASTIC PROBLEMS FOR ANISOTROPIC MEDIA

BY

AVTAR SINGH, *Kharagpur*

In this report three methods of solving thermoelastic problems for antisotropic media were discussed. The first method (Kaliski, 1957) is perfectly general and applicable to a body of any shape with the most general type of anisotropy. In the equation of heat conduction Biot's coupling term (giving the effect of strain on temperature) as well as the time dependency of temperature and displacements can be taken into account. The three equations of small motions and the equation of heat conduction can be written in terms of displacement (u_1, u_2, u_3) and the temperature (u_4) as follows:

$$L_{ik}(u_k) = P_i \ (i, k = 1, 2, 3, 4, k \text{ summed}) \qquad \ldots \quad (1)$$

where $P_{1,2,3}$ are the body force components per unit volume and P_4 is the rate of internal heat production per unit volume, and L_{ik} are certain linear operators. If the stress functions $\psi_{1,2,3}$ and the temperature function ψ_4 are introduced by the equations

$$u_1 = \begin{vmatrix} \psi_1 & L_{21} & L_{31} & L_{41} \\ \psi_2 & L_{22} & L_{32} & L_{42} \\ \psi_3 & L_{23} & L_{33} & L_{43} \\ \psi_4 & L_{24} & L_{34} & L_{44} \end{vmatrix}, \quad u_2 = \begin{vmatrix} L_{11} & \psi_1 & L_{31} & L_{41} \\ L_{12} & \psi_2 & L_{32} & L_{42} \\ L_{13} & \psi_3 & L_{33} & L_{43} \\ L_{14} & \psi_4 & L_{34} & L_{44} \end{vmatrix} \text{ etc.} \qquad \ldots \quad (2)$$

then ψ_i are required to satisfy the equations

$$\begin{vmatrix} L_{11} & L_{21} & L_{31} & L_{41} \\ L_{12} & L_{22} & L_{32} & L_{42} \\ L_{13} & L_{23} & L_{33} & L_{43} \\ L_{14} & L_{24} & L_{34} & L_{44} \end{vmatrix} (\psi_i) = P_i \qquad \ldots \quad \ldots \quad (3)$$

For special types of anisotropy and isotropy these equations reduce to the standard known equations.

The second method (Sharma, 1958) which is applicable to three-dimensional steady state stress distribution in transversely isotropic semi-infinite media is given by Sharma in terms of two stress functions $\varphi_{1,2}$ and a temperature function $\bar{T}$. Functions $\varphi_{1,2}$ satisfy the same second order partial differential equations as those given by Elliott (1948).

The third method is applicable to any axi-symmetrical steady state stress distribution problem from transversely isotropic medium. In this method (Singh) the stresses and the displacements can be expressed in terms of two stress functions $\varphi_{1,2}$ which satisfy the differential equations

$$\left.\begin{aligned} \nabla_1^2 \varphi_1 + K_{22}^2 \frac{\partial^2 \varphi_1}{\partial z^2} &= \frac{1}{a}\left(\frac{\partial \varphi_2}{\partial z^2} - c\,T\right) \\ \nabla_1^2 \varphi_2 + K_{21}^2 \frac{\partial^2 \varphi_2}{\partial z^2} &= a_1\,T \end{aligned}\right\} \quad \ldots \quad (4)$$

where,

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

and K_{22}^2, K_{21}^2, a, c and a, are certain constants depending upon the elastic and thermal properties of the material, T is the temperature and (r, θ, z) are the cylindrical polar coordinates.

DEPARTMENT OF APPLIED MATHEMATICS,
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FINITE STRAIN THEORY

By

P. P. CHATTARJI, *Calcutta*

Lagrangian and Eulerian Co-ordinates. In Lagrangian co-ordinates the strain tensor η_{ij} is given by

$$2\eta_{ij} = u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \ (i, j, k = 1, 2, 3) \tag{1}$$

where u_1, u_2, u_3 are the displacements. In Eulerian co-ordinates the strain-tensor ϵ_{ij} is given by

$$2\epsilon_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}, \ (i, j, k = 1, 2, 3) \tag{2}$$

Since the boundary displacements or the boundary tractions are known in the strained state and the stress tensor τ_{ij} is defined in terms of Eulerian co-ordinates the strain tensor must also be referred to the same system.

Body Stress Equations of Equilibrium. The stress tensor τ_{ij} satisfies

$$\tau_{ij,i} + F_j = 0 \tag{3}$$

Consistency Equations. The vanishing of Schwarz-Christoffel tensor is a necessary and sufficient condtion for flat space time.

Hence $$\{\mu\sigma, \alpha\}[\alpha\gamma, \epsilon] - \{\mu\gamma, \alpha\}[\alpha\sigma, \epsilon] - \frac{\partial}{\partial x_\gamma}[\mu\sigma, \epsilon] - \frac{\partial}{\partial x_\sigma}[\mu\gamma, \epsilon] = 0 \tag{4}$$

In three dimensions (4) has only six components. The vanishing of these contitutes the six consistency equations. Little use can be made of these equations because of their complicated nature.

Stress-strain Relation. Any assumption that is made regarding the stress-strain relation must be such that these relations are invariant for all transformations from one set of orthogonal axes to another. In other words these relations must be in tensor form.

Seth (1935) assumed that if terms only of the second degree in the invariants I_1 and I_2 in the strain energy function W be retained, the stress-strain tensor relation is linear and is as follows

$$\tau_{ij} = \lambda\delta_{ij}I_1 + 2\mu\epsilon_{ij} \tag{5}$$

where λ and μ are Lamé's constants and δ_{ij} is the Kronecker delta. Seth's theory has since been known as finite deformation theory.

Murnaghan (1937) assumed that at least third degree terms must be kept in W. If l, m, n are the new elastic constants

$$\varrho_0 W = \tfrac{1}{2}(\lambda + 2\mu)I_1^2 - 2\mu I_2 + lI_1^3 + mI_1I_2 + nI_3 \tag{6}$$

ϱ_0 being the density in the unstrained state. If ϱ is the density in the strained state

$$\varrho/\varrho_0 = (1 - 2I_1 + 4I_2 - I_3)^{\frac{1}{2}} = 1 - I_1 + \ldots \tag{7}$$

Stresses are derived from (6) by the formula

$$\tau_{ij} = \varrho\left[\frac{\partial W}{\partial \varepsilon_{ij}} - 2\varepsilon_{\alpha i}\frac{\partial W}{\partial \varepsilon_{\alpha j}}\right] \tag{8}$$

Applications. Seth has given a number of applications of his theory. He has discussed simple tension, pure flexure, torsion and flexure etc. (See Seth (1950)). He has also given a law of elastic failure

$$p_1 - p_3 + k(p_1 + p_2 + p_3) = \text{constant} \tag{9}$$

where p_1, p_2, p_3 are the principal stresses in descending order of magnitude,

Murnaghan (1951) has discussed (*i*) simple shear and tension, (*ii*) compression of a spherical shell and circular tube, (*iii*) torsion of a circular cylinder.

Chattarji (1953) has discussed finite deformation in the interior of the earth using Seth's theory.

Various other theories have been proposed by a number of authors which, however, cannot be discussed in this short space. A comprehensive literature has been given by Truesdell (1952).

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POROUS ELASTIC MEDIUM

BY

GUNADHAR PARIA, *Kharagpur*

1. Introduction. The deformation of a porous elastic medium containing a viscous compressible fluid has been the subject of much investigation, an extensive account of which may be found in recent papers by Biot (1955, 1956), Lubinsky (1955) and Paria (1957, 1958*a*, 1958*b*). In such a medium we consider a solid-fluid system in which the solid skeleton with statistical distribution of inter-connected pores is assumed to be purely elastic while the pores are filled, with fluids compressible or incompressible, viscous or non-viscous. The seepage of water from a mass of clay may be cited as an example. The porosity f of the medium is defined as $f = V_p/V$ where V_p is the volume of the inter-connected pores contained in a sample of bulk volume V. An equivalent definition is $f = S_p/S$ where S_p is the inter-connected pore area in a cross-sectional area S of the bulk material. The stress tensor in the medium considered is

$$\begin{pmatrix} \sigma_{xx}+\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+\sigma \end{pmatrix}$$

with the symmetric property $\sigma_{ij} = \sigma_{ji}$ where σ_{ij} is the stress on the solid skeleton and σ is the fluid stress defined as

$$\sigma = -fp$$

p being the pressure of the fluid. If the average displacement components of the solid be denoted by $\vec{u}(u_x, u_y, u_z)$ and that of the fluid by $\vec{U}(U_x, U_y, U_z)$ then, the strain and dilation components for solid and the fluid respectively are

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}, \quad e = \operatorname{div} \vec{u}, \text{ etc.}$$

and

$$\varepsilon_{xx} = \frac{\partial U_x}{\partial x}, \quad \varepsilon_{xy} = \frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial y}, \quad \varepsilon = \operatorname{div} \vec{U}, \text{ etc.}$$

2. Fundamental Equations. Similar to Hooke's law, it is assumed that the seven stress components (σ_{ij}, σ) are linear functions of of the seven strain components (e_{ij}, ε). Thus we have

$$\begin{bmatrix} \\ \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \cdots\cdots\cdots c_{17} \\ c_{21}, & c_{22} \cdots\cdots\cdots c_{27} \\ \cdots & \cdots\cdots\cdots\cdots \\ \cdots & \cdots\cdots\cdots\cdots \\ & \\ \cdots & \cdots\cdots\cdots\cdots \\ c_{71}, & c_{72}, \cdots\cdots\cdots c_{77} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ \epsilon \end{bmatrix}$$

where the property $c_{ij} = c_{ji}$ follows from the assumption of the existence of an elastic potential. The flow of fluid is governed by the generalised form of Darcy's law as

$$\begin{bmatrix} \dfrac{\partial\sigma}{\partial x} + \varrho_1 X \\ \dfrac{\partial\sigma}{\partial y} + \varrho_1 Y \\ \dfrac{\partial\sigma}{\partial z} + \varrho_1 Z \end{bmatrix} = \begin{bmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{bmatrix} \begin{bmatrix} \dot{U}_x - \dot{u}_x \\ \dot{U}_y - \dot{u}_y \\ \dot{U}_z - \dot{u}_z \end{bmatrix}$$

where $\varrho_1 = f\varrho'$ (ϱ' being the mass density of the fluid), (X, Y, Z) are the components of the body force per unit mass of the bulk material and a dot denotes differentiation with respect to time t. The symmetry of the coefficients

$$b_{ij} = b_{ji}$$

results from the assumption of the existence of a dissipation function. In the absence of the body force (X, Y, Z), Darcy's law in the most simple case states that the velocity (here the relative velocity) of the fluid is proportional to the pressure gradient. The equations of equilibrium (considering only the quasi-static theory) for the bulk material are

$$\frac{\partial(\sigma_{xx}+\sigma)}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} + \varrho X = 0$$

and similar other two equations. Here ϱ is the mass density of the bulk material. The above three sets of equations with appropriate boundary conditions on the solid stresses (or displacements) and the fluid stress (or displacements) together with compatibility equations among the stress (or strain) components determine the solution.

3. Isotropic Medium. If the material is assumed to be completely isotropic the stress-strain relations reduce to

$$\begin{aligned}\sigma_{xx} &= 2Ne_{xx}+Ae+Q\varepsilon\\ \sigma_{yy} &= 2Ne_{yy}+Ae+Q\varepsilon\\ \sigma_{zz} &= 2Ne_{zz}+Ae+Q\varepsilon\\ \sigma_{yz} &= Ne_{yz},\ \sigma_{zx} = Ne_{zx},\ \sigma_{xy} = Ne_{xy}\\ \sigma &= Qe+R\varepsilon\end{aligned}$$

while Darcys's law simplifies to

$$\frac{\partial\sigma}{\partial x}+\varrho_1 X = b(\dot{U}_x-\dot{u}_x)$$

$$\frac{\partial\sigma}{\partial y}+\varrho_1 Y = b(U_y-\dot{u}_y)$$

$$\frac{\partial\sigma}{\partial z}+\varrho_1 Z = b(\dot{U}_z-\dot{u}_z)$$

The four material constants A, N, Q and R are characteristic of the solid-fluid system whereas b determines the permeability of the solid with respect to the given fluid. In fact, the constant N is the shear modulus while A, Q and R may be expressed in terms of jacketed compressibility, unjacketed compressibility and the fluid content factor.

4. General Solution. In two-dimensional or axially symmetric problems the general solution may be obtained in a convenient form although a formal solution in three dimensions is not altogether ruled out. In axially symmetric problems it has been shown by Paria (1958*b*) that the stresses and displacements can be expressed in terms of a stress function φ and the fluid stress σ

where $$\nabla^6\varphi = L\nabla^4\frac{\partial\varphi}{\partial t}$$

and $$\nabla^6\varphi = L\nabla^2\frac{\partial\sigma}{\partial t}$$

L being a constant expressible in terms of the material constants. The above equations show that the function φ is the sum of the solutions of the biharmonic equation

$$\nabla^4\varphi = 0$$

and the heat conduction equation

$$\nabla^2\varphi = L\frac{\varphi}{\partial t}$$

whereas σ is the sum of the solutions of the harmonic equation

$$\nabla^2\sigma = 0$$

and the heat conduction equation

$$\nabla^2\sigma = L\frac{\sigma}{\partial t}$$

As an illustration, the stress distribution in a semi-infinite medium under a surface load applied suddenly was considered in the same paper. The transient solution for the problem was obtained in terms of error functions.

It is to be observed that the complete solution of a particular problem is, in general, involved because of the fact that it is the combination of the solutions of three different problems, namely, the harmonic, the biharmonic and the heat conduction problems.

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HYPO-ELASTICITY

BY

P. D. S. VERMA, *Kharagpur*

In rational mechanics, a lot of thinking is done as to how the various materials respond to a given set of forces and this response must necessarily be independent of the observer. The problem is how to limit our ideal materials to those whose response is properly invariant with respect to the change of the observer. In hypo-elasticity, a very recent subject, we study the elastic response by formulating it in terms of rates. This differs from older theories in that as aquations are of a form admissible for deformations of any magnitude and speed. The form of the stress-strain relations is not known in advance, rather it depends on the problem. It offers a mechanical theory including plastic flow not as a primary condition but as the result of previous circumstances leading upto elastic failure. Arbitrary initial stresses are possible to take into account. The constitutive equations of this theory are

$$\tilde{S} = \delta g_0 1 + g_1 d + \delta g_2 S + M g_3 1 + \tfrac{1}{2} g_4 (dS + Sd) + \delta g_5 S^2 + M g_6 S + N g_7 1 + \tfrac{1}{2} g_8 (dS^2 + S^2 d) \\ + M g_9 S^2 + N g_{10} S + N g_{11} S^2,$$

where
$$\tilde{S} = \frac{\partial}{\partial t} S^{ij} + S^{ij}{}_{,k} v^k - S^{ik} v^j{}_{,k} - S^{kj} v^i{}_{,k} + S^{ij} v^k{}_{,k},$$

$S \equiv \|$stress tensor$\|$, $d \equiv \|$rate of deformation tensor$\|$, g's are the functions of the pricipal invariants of S,

$$\delta = d^k{}_k, \qquad M \equiv S^i{}_j d^j{}_i, \qquad N \equiv S^i{}_j S^j{}_k d^k{}_i,$$

together with these equations, the usual equations of motion and mass, in consistence with given boundary conditions, constitute the basic equations of an hypo-elastic problem.

Further it is possible, for some hypo-elastic materials, to introduce a scalar potential or the strain energy function of elastic potential or the strain energy function of elasticity.

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Note.—Abstracts of the talks by Dr. D. N. Mitra and Sri S. Dutta were not recived.

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THE ANALYSIS OF NORMALLY LOADED THIN ISOTROPIC PLATES WITH FREE EDGES

By

W. A. Bassali, *Alexandria, Egypt.*

(Communicated by the Editorial Secretary—Received—March 5, 1960)

Muskhelishvili's complex variable method of solving the biharmonic equation is applied in this paper to establish exact solutions in closed forms for a finite circular plate, an infinite plate with a circular hole and a half-plane when these plates are acted upon by certain line loadings distributed along the circumference of a circle and the edges of the plates are free. The principle of superposition can then be used to obtain the deflexions, moments and shears when any of these plates is supported at several interior or boundary points and is subjected to general distributions of normal thrust over the area of a circle. As an application plates having the shape of circular sectors simply supported along the radial edges, free along the circular edge and normally loaded over the area of a concentric sector are analysed.

1. **Introduction.** Methods using complex variables have been applied to isotropic and anisotropic plate problems by many authors, references are given by Muskhelishvili (1949) and Sokolnikoff (1942). Some flexural problems of thin plates under lateral loading have been solved by Yu (1956). The three usual types of boundary conditions for clamped, simply supported and free edges are formulated and various kinds of normal loading, both concentrated and distributed, are discussed. Problems concerning the transverse flexure of an elastically restrained isotropic circular plate have been considered by Bassali and Dawoud (1956) and the author (1956) when the plate is subjected to the loadings $p_0R^m \cos n\Theta$ $(n = 0, n = 1, n > 1)$ over an eccentric circular patch where R and Θ are measured from the centre of the patch and the common diameter of plate and patch. The corresponding problems for a circular plate with a free boundary and supported at any number of interior or boundary points have been solved by the author (1957*b*, 1958*a*) only when the plate is acted upon by the two types of loading $p = p_0R^m$ or $p = p_0R^m \cos(\Theta - \alpha)$ over the area of the eccentric circle, α being a constant angle. The case of concentrated couples is included as a limiting case of the latter type of loading.

Solutions appropriate to an infinite plate elastically restrained at an inner circular boundary and subjected to the three types of loading $p = p_0R^m \cos(n\Theta - \alpha)$ $(n = 0, n = 1, n > 1)$ distributed over the area of a circle have been obtained in a series of papers [Bassali (1957*a*,*d*) and Bassali and Nassif (1958)]. When the circular hole in the infinitely large plate is free and the plate is supported at several points and is under the load p_0R^m or $p_0R^m \cos(\Theta - \alpha)$ over the area of a circle the corresponding solutions have also been found (Bassali 1957*c*, 1958*b*) and particular examples have been worked out in detail.

To complete this series of solutions there remains now the two cases of a circular plate with a free edge or an infinite plate containing a free circular hole when either plate is acted upon by the lateral load $p = p_0 R^m \cos(n\Theta-\alpha)$ $(n>1)$ spread over an arbitrary circular domain. It is with these two problems that we are going to deal in this paper. The solution for a half-plane bent by the same pressure is then derived as a limiting case of either of these problems It is to be noticed that this type of loading (corresponding to $n>1$) keeps the plate in equilibrium without the requirement of interior or boundary supports. Since the cases $n=0, 1$ which require separate treatments have been studied before (Bassali 1957*b*, *c*; 1959*a*, *b*) it follows that this paper completes the solution of the problems of a finite circular plate with a free edge or an infinite plate with a free circular hole when either plate is supported at any number of points and the intensity of normal thrust over an arbitrary circular domain is expressed as a Fourier-Taylor expansion

$$p = \sum_{m,n} R^m \left(a_{mn}\cos n\Theta + b_{mn}\sin n\Theta\right).$$

Results obtained for the circular plates are applied to sectorial plates simply supported along the straight edges, free along the circular edge and subject to certain loadings over the area of a concentric sector.

2. **Fundamental Results.** When the middle plane of a transversely loaded thin isotropic plate bounded by a closed contour C is taken in the complex plane of $z = x+iy = re^{i\theta}$, the differential equation satisfied by the deflexion w in a region free from lateral load has the form

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{16\partial^4 w}{\partial z^2\partial \bar{z}^2} = 0, \tag{2.1}$$

bars being used to indicate conjugate complex quantities. The general solution of this biharmonic equation may be represented in terms of two complex potentials $\varphi(z)$ and $\chi(z)$

$$w = 2\,\mathbf{Re}[\bar{z}\,\varphi(z) + \chi(z)], \tag{2.2}$$

where w is measured positively downwards in a direction perpendicular to the plate and **Re** stands for the real part. Adopting Timoshenko's notation (1940), the bending, twisting moments and shearing forces at any point $z = re^{i\theta}$ of the unloaded region are determined by

$$M_r + M_\theta = -8D(1+\nu)\mathbf{Re}\,\varphi'(z), \tag{2.3a}$$

$$M_r - M_\theta + 2iM_{r\theta} = -4D(1-\nu)\left[z\varphi''(z) + (z/r)^2\,\psi'(z)\right], \tag{2.3b}$$

$$Q_r - iQ_\theta = -8D(z/r)\,\varphi''(z), \tag{2.3c}$$

where

$$\psi(z) = \chi'(z), \tag{2.4}$$

$$D = Eh^3/12(1-\nu^2), \tag{2.5}$$

E, ν and h are respectively, Young's modulus, Poisson's ratio and thickness of the plate.

We assume that the plate is subjected to a variable transverse line loading of intensity p extending along a smooth closed curve Γ which lies totally inside the plate and divides it into the two regions 1 inside Γ and 2 between Γ and C. By considering line loadings we avoid the use of particular integrals required when the lateral loading is distributed over a certain domain. It was shown (Bassali and Nassif, 1958) that the transition conditions to be satisfied across Γ may be written as

$$[\bar{z}\varphi(z)+z\,\bar{\varphi}(\bar{z})+\chi(z)+\bar{\chi}(\bar{z})]_1^2=0, \tag{2.6a}$$

$$[\bar{z}\,\varphi'(z)+\bar{\varphi}(\bar{z})+\psi(z)]_1^2=0, \tag{2.6b}$$

$$[\varphi'(z)+\bar{\varphi}'(\bar{z})]_1^2=0, \tag{2.6c}$$

$$[\varphi''(z)]_1^2=pe^{-i\beta}/8D, \tag{2.6d}$$

where $\varphi_1(z)$, $\chi_1(z)$; $\varphi_2(z)$, $\chi_2(z)$ are the complex potentials in regions 1, 2, respectively, and β is the angle made with the x-axis by the normal n at the point of Γ under consideration.

The boundary condition for plate problems in which the bending moments and shear resultants are prescribed has been obtained by Lechnitzky (1938) whose result for a free boundary C takes the form

$$\kappa\varphi_2(z)+z\bar{\varphi}'_2(\bar{z})+\bar{\psi}_2(\bar{z}) = A'+iB'z \text{ on } C, \tag{2.7}$$

where

$$\kappa = (3+\nu)/(\nu-1), \tag{2.8}$$

A' is a complex constant and B' is a real constant, both arbitrary.

3. Thin Circular Plate under a General Line Loading along an Eccentric Circle. Consider a thin circular plate of centre O, radius c and free boundary C. Let Γ denote the boundary of an eccentric circle with centre $Q_0(z_0 = r_0e^{i\theta_0})$ and radius $b \leq c-r_0$. We assume that the plate is subject to a lateral line loading along Γ whose intensity at any point T is given by

$$p = p_0\cos(k\Theta-\alpha)\ (k>1), \tag{3.1}$$

where p_0, α are constants, k is an integer and Θ is the angle which TQ_0 make with the positive direction of the x-axis. [See Fig. 1, p. 268 of Bassali (1958a)]. It is easily shown that the plate is in equilibrium under the load (3.1). The problem to be solved now is to determine the complex potentials $\varphi_1(z)$, $\chi_1(z)$; $\varphi_2(z)$, $\chi_2(z)$ which fulfil the boundary condition (2.7) on C and satisfy the continuity conditions (2.6) corresponding to the load intensity (3.1) along Γ. For all points on Γ we have

$$Z\bar{Z}= b^2, \tag{3.2a}$$

where

$$Z = z-z_0. \tag{3.2b}$$

From (3.1) and (3.2) it follows that

$$p = \tfrac{1}{2}p_0(\bar{\lambda}U^k+\lambda U^{-k}), \tag{3.3}$$

where

$$\lambda = e^{i\alpha},\ U = Z/b. \tag{3.4}$$

Substituting from (3.3) in (2.5d) and noting that $\beta = \Theta$ we find

$$[\varphi''(z)]_1^2 = \delta(\lambda U^{k-1}+\lambda U^{-k-1}), \tag{3.5}$$

where

$$\delta = p_0/16D. \tag{3.6}$$

We now assume that the continuity conditions (2.6) are satisfied by

$$[\varphi]_1^2 = Z(AU^k+BU^{-k}),\ [\chi+\bar{z}_0\varphi]_1^2 = CU^k+EU^{-k}, \tag{3.7}$$

where A, B, C and E are, in general, complex constants to be found. Introducing (3.7) in (3.5), (2.6b) and equating the coefficients of the same powers of Z on both sides of each equation lead to

$$A = \frac{\delta\bar{\lambda}b}{k(k+1)},\ B = \frac{\delta\lambda b}{k(k-1)},\ C = -\frac{\delta\bar{\lambda}b^3}{k(k-1)},\ E = -\frac{\delta\lambda b^3}{k(k+1)},$$

and hence

$$[\varphi]_1^2 = \frac{\delta bZ}{k}\left[\frac{\bar{\lambda}U^k}{k+1}+\frac{\lambda U^{-k}}{k-1}\right], \tag{3.8a}$$

$$[\chi]_1^2 = -\frac{\delta b}{k}\left[\left(\frac{\bar{z}_0Z}{k+1}+\frac{b^2}{k-1}\right)\bar{\lambda}U^k+\left(\frac{\bar{z}_0Z}{k-1}+\frac{b^2}{k+1}\right)\lambda U^{-k}\right]. \tag{3.8b}$$

It is now easily checked that all the transition conditions (2.6) are satisfied by (3.8). Since φ_1 and χ_1 must be regular at $z = z_0 (Z = 0)$ it follows that the terms containing U^{-k} in (3.8a), (3.8b) must enter in φ_2, χ_2, respectively. The four complex potentials φ_1, χ_1, φ_2 and χ_2 may thus be taken in the forms

$$\varphi_1 = -\frac{\delta b^2\bar{\lambda}}{k(k+1)}U^{1+k}+\varphi_0(z), \tag{3.9a}$$

$$\chi_1 = \frac{\delta b\bar{\lambda}}{k}\left(\frac{\bar{z}_0Z}{k+1}+\frac{b^2}{k-1}\right)U^k+\chi_0(z) \tag{3.9b}$$

$$\varphi_2 = \frac{\delta b^2\lambda}{k(k-1)}U^{1-k}+\varphi_0(z), \tag{3.9c}$$

$$\chi_2 = -\frac{\delta b\lambda}{k}\left(\frac{\bar{z}_0Z}{k-1}+\frac{b^2}{k+1}\right)U^{-k}+\chi_0(z), \tag{3.9d}$$

where $\varphi_0(z)$ and $\chi_0(z)$ are functions of z which are regular over the entire plate and will now be determined from the boundary condition (2.7). Substituting from (3.9c), (3.9d) in (2.7) and writing $z = c\zeta$, $z_0 = c\zeta_0$,

$$\Phi_0(\zeta) = \varphi_0(z),\ \Psi_0(\zeta) = \psi_0(z) = X'_0(z), \tag{3.10}$$

we find that the boundary condition is transformed to

$$\kappa\Phi_0(\sigma) + \sigma\bar{\Phi}'_0(\bar{\sigma}) + \bar{\Psi}_0(\bar{\sigma}) = A' + icB'\sigma + f(\sigma), \tag{3.11}$$

where $\sigma = e^{i\theta}$ is any point on the boundary γ of the unit circle and

$$f(\sigma) = \delta b^2\Big[\bar{\lambda}\Big(\frac{q\sigma}{1-\bar{\zeta}_0\sigma}\Big)^k\Big\{\frac{\sigma-\zeta_0}{kq} - \frac{q\sigma}{(k+1)(1-\bar{\zeta}_0\sigma)}\Big\} - \frac{\kappa\lambda}{k(k-1)}\Big(\frac{q}{\sigma-\zeta_0}\Big)^{k-1}\Big]\Big(q = \frac{b}{c}\Big). \tag{3.12}$$

Following Muskhelishvili's procedure of multiplying (3.11) and its conjugate throughout by $\dfrac{1}{2\pi i}\dfrac{d\sigma}{\sigma-\zeta}$ and integrating around γ we find, apart from a constant in $\Phi_0(\zeta)$ or $\Psi_0(\zeta)$,

$$\Phi_0(\zeta) = \frac{1}{2\pi i\kappa}\int_\gamma \frac{f(\sigma)}{\sigma-\zeta}\,d\sigma - \frac{\alpha_1}{\kappa}\zeta, \tag{3.13a}$$

$$\Psi_0(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\bar{f}(\sigma^{-1})}{\sigma-\zeta}\,d\sigma + \frac{\alpha_1 - \Phi'_0(\zeta)}{\zeta}. \tag{3.13b}$$

where

$$\alpha_1 = \frac{1}{\kappa+1}\Big[\frac{d}{d\zeta}\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)}{\sigma-\zeta}\,d\sigma\Big)\Big]_{\zeta=0} \tag{3.13c}$$

is the real coefficient of ζ in the expansion of $\Phi_0(\zeta)$ as power series of positive powers of ζ. [See Yu's paper (1954), p. 131]. When the integrals are evaluated by residue theorems it is found that $\alpha_1 = 0$ and

$$\Phi_0(\zeta) = \frac{\delta\bar{\lambda}bc}{\kappa}\Big[\frac{u_0}{k} - \frac{q^2\zeta}{(k+1)v_0}\Big]\Big(\frac{q\zeta}{v_0}\Big)^k + C, \tag{3.14a}$$

$$\Psi_0(\zeta) = -\frac{\delta\bar{\lambda}bc}{k\kappa\zeta}\Big[1 + \frac{\kappa^2 v_0}{k-1} + \frac{ku_0}{\zeta v_0} - \frac{k\,q^2}{v_0^2}\Big]\Big(\frac{q\zeta}{v_0}\Big)^k,$$

where C is, in general, a complex constant and

$$u_0 = \zeta - \zeta_0,\ v_0 = 1 - \bar{\zeta}_0\zeta. \tag{3.15}$$

Introducing (3.14) into (3.9), making use of (3.10) and (2.4), we find that the complete solution of the problem is furnished by

$$\varphi(z) = \frac{\delta b}{k}\Big[\frac{c\bar{\lambda}}{\kappa}\Big\{u_0 - \frac{kq^2\zeta}{(k+1)v_0}\Big\}\Big(\frac{q\zeta}{v_0}\Big)^k + F(Z)\Big] + C, \tag{3.16a}$$

$$\psi(z) = -\frac{\delta b}{k}\Big[\frac{c\bar{\lambda}}{\kappa\zeta}\Big\{1 + \frac{\kappa^2 v_0}{k-1} + \frac{ku_0}{\zeta v_0} - \frac{kq^2}{v_0^2}\Big\}\Big(\frac{q\zeta}{v_0}\Big)^k + G(Z)\Big], \tag{3.16b}$$

where
$$\left.\begin{aligned} F(Z) &= -\frac{\bar{\lambda} Z}{k+1}\left(\frac{Z}{b}\right)^{k} \text{ in region 1,} \\ F(Z) &= \frac{\lambda Z}{k-1}\left(\frac{b}{Z}\right)^{k} \text{ in region 2,} \end{aligned}\right\} \tag{3.17}$$

$$\left.\begin{aligned} G(Z) &= -\frac{\bar{\lambda}}{Z}\left(\bar{z}_0 Z + \frac{kb^2}{k-1}\right)\left(\frac{Z}{b}\right)^{k} \text{ in region 1,} \\ G(Z) &= -\frac{\lambda}{Z}\left(\bar{z}_0 Z + \frac{kb^2}{k+1}\right)\left(\frac{b}{Z}\right)^{k} \text{ in region 2.} \end{aligned}\right\} \tag{3.18}$$

Since the solution for a thin circular plate with a free edge and acted upon normally by the system of m forces N_s at the points $z_s = c\zeta_s$ $(s = 1, 2, \ldots, m)$ has been derived before (Bassali 1957*b*, 1958*a*) it follows, by applying the principle of superposition, that the solution appropriate to the same plate supported at the same points and subject to the load (3.1) along Γ is given by

$$\varphi(z) = \frac{c}{16\pi\kappa D}\sum_{s=1}^{m} N_s\left\{u_s\,(\kappa \log u_s - \log v_s) - \frac{1-\varrho_s^2}{\kappa+1}\,\bar{\zeta}\right\}$$
$$+\frac{p_0 bc}{16kD}\left[\frac{\bar{\lambda}}{\kappa}\left\{u_0 - \frac{kq^2\zeta}{(k+1)v_0}\right\}\left(\frac{q\zeta}{v_0}\right)^{k} + F(Z)\right] + C, \tag{3.19a}$$

$$\psi(z) = -\frac{c}{16\pi\kappa D}\sum_{s=1}^{m} N_s\left\{\bar{\zeta}_s\left(\kappa \log u_s - \log v_s + \frac{1-\varrho_s^2}{v_s}\right) + \frac{\kappa^2-1}{\zeta}\,v_s \log v_s\right\}$$
$$-\frac{p_0 bc}{16kD}\left[\frac{\bar{\lambda}}{\kappa\zeta}\left\{1 + \frac{\kappa^2 v_0}{k-1} + \frac{ku_0}{\zeta v_0} - \frac{kq^2}{v_0^2}\right\}\left(\frac{q\zeta}{v_0}\right)^{k} + G(Z)\right], \tag{3.19b}$$

where $\qquad \varrho = |\zeta|,\ \varrho_s = |\zeta_s|,\ u_s = \zeta - \zeta_s,\ v_s = 1 - \bar{\zeta}_s\zeta\ (s = 0, 1, 2, \ldots, m).$ $\qquad$ (3.20)

The moments and shears at any point of the plate can now be obtained by substitution from (3.16), (3.17) and (3.18) in (2.8). In region 2 we get

$$\begin{matrix} M_r \\ M_\theta \end{matrix} = \mp\frac{1}{8}(1-\nu)p_0 b\,\mathbf{Re}\,\bar{\lambda}\left[\left(\frac{q}{\bar{u}_0}\right)^{k}\left\{\frac{\varrho^2}{\varrho^2-\bar{\zeta}_0\zeta}\left(\frac{u_0}{\zeta} - \frac{q^2}{\varrho^2-\bar{\zeta}_0\zeta}\right) \pm \frac{1+\kappa}{k}\right\} + \frac{(q\zeta/v_0)^k}{\kappa v_0}\left\{\frac{1}{\varrho^2}\left(\frac{u_0}{\zeta} - \frac{q^2}{v_0}\right.\right.\right.$$
$$\left.\left.\left. + \frac{1-\kappa^3}{k}\,v_0\right) + \left(1 - \frac{1}{\varrho^2}\right)\left(2\,\frac{1-\varrho_0^2+q^2}{v_0} + \frac{k-1}{\zeta v_0}u_0 - \frac{k+2}{v_0^2}q^2\right) \mp (1+\kappa)\left(\frac{u_0}{\zeta} - \frac{q^2}{v_0} + \frac{v_0}{k}\right)\right\}\right], \tag{3.21a}$$

$$M_{r\theta} = -\frac{1}{8}(1-\nu)p_0 b\,\mathbf{Im}\,\bar{\lambda}\left[\frac{\varrho^2}{\varrho^2-\bar{\zeta}_0\zeta}\left(\frac{q}{\bar{u}_0}\right)^{k}\left(\frac{q^2}{\varrho^2-\bar{\zeta}_0\zeta} - \frac{u_0}{\zeta}\right)\right.$$
$$\left. + \frac{(q\zeta/v_0)^k}{\kappa v_0}\left\{\frac{1}{\varrho^2}\left(\frac{u_0}{\zeta} - \frac{q^2}{v_0} + \frac{1-\kappa^2}{k}\,v_0\right) + \left(1 - \frac{1}{\varrho^2}\right)\left(2\,\frac{1-\varrho_0^2+q^2}{v_0} + \frac{k-1}{\zeta v_0}u_0 - \frac{k+2}{v_0^2}q^2\right)\right\}\right], \tag{3.21b}$$

$$Q_r - iQ_\theta = -\frac{p_0 b}{2\kappa r}\left[\frac{\lambda\kappa\zeta}{u_0}\left(\frac{q}{u_0}\right)^{k} + \frac{\bar{\lambda}(q\zeta/v_0)^k}{v_0^2}\left\{2(1-\varrho_0^2+q^2) + \frac{k-1}{\zeta}u_0 - \frac{k+2}{v_0}q^2\right\}\right], \tag{3.21c}$$

Setting $\varrho = 1$, $\zeta = \sigma$, $\bar{\zeta} = \sigma^{-1}$ in (3.21*a*) we easily see that $M_r = 0$ on C, as it should. Considerable simplifications in the foregoing results occur when the load (3.1) is distributed along the circumference of a circle concentric with the plate. Putting $\zeta_0 = 0$ in (3.19) and (3.21) yields

$$\varphi_1 = \delta\bar{\lambda}b^2\left[\frac{\zeta^2}{\kappa}\left(\frac{1}{k} - \frac{q^2}{k+1}\right)(q\zeta)^{k-1} - \frac{1}{k(k+1)}\left(\frac{\zeta}{q}\right)^{k+1}\right], \tag{3.22a}$$

$$\psi_1 = \delta\bar{\lambda}b^2\left[\frac{q}{(k-1)\zeta}\left(\frac{\zeta}{q}\right)^{k} - \frac{1}{\kappa}\left(1 + \frac{1}{k} - q^2 + \frac{\kappa^2}{k^2-k}\right)(q\zeta)^{k-1}\right], \tag{3.22b}$$

$$\varphi_2 = \delta b^2\left[\frac{\lambda}{k(k-1)}\left(\frac{q}{\zeta}\right)^{k-1} + \frac{\bar{\lambda}\zeta^2}{k}\left(\frac{1}{k} - \frac{q^2}{k+1}\right)(q\zeta)^{k-1}\right], \tag{3.23a}$$

$$\psi_2 = \delta b^2\left[\frac{\lambda q}{(k+1)\zeta}\left(\frac{q}{\zeta}\right)^{k} - \frac{\bar{\lambda}}{\kappa}\left(1 + \frac{1}{k} - q^2 + \frac{\kappa^2}{k^2-k}\right)(q\zeta)^{k-1}\right]; \tag{3.23b}$$

$$\begin{matrix} M_r \\ M_\theta \end{matrix} = \mp\frac{1-\nu}{8\kappa}p_0 b q^k\left[\kappa\varrho^{-k}\left(1 - \frac{q^2}{\varrho^2} \pm \frac{1+\kappa}{k}\right) + \varrho^k\left\{(1+k-kq^2)\left(1 \mp \frac{1+\kappa}{k} + \frac{1-k}{k\varrho^2}\right) - \frac{\kappa^2}{k\varrho^2}\right\}\right]\cos(k\theta - \alpha), \tag{3.24a}$$

$$M_{r\theta} = -\frac{1-\nu}{8\kappa}p_0 b q^k\left[\kappa\varrho^{-k}\left(\frac{q^2}{\varrho^2} - 1\right) + \varrho^k\left\{(1+k-kq^2)\left(1 + \frac{1-k}{k\varrho^2}\right) - \frac{\kappa^2}{k\varrho^2}\right\}\right]\sin(k\theta - \alpha), \tag{3.24b}$$

$$Q_r = -\frac{p_0 q^{k+1}}{2\kappa\varrho}\left[\varrho^k(1+k-kq^2) + \kappa\varrho^{-k}\right]\cos(k\theta - \alpha), \tag{3.24c}$$

$$Q_\theta = \frac{p_0 q^{k+1}}{2\kappa\varrho}\left[\varrho^k(1+k-kq^2) - \kappa\varrho^{-k}\right]\sin(k\theta - \alpha). \tag{3.24d}$$

From (3.24*b*) and (3.24*c*) it is easily checked that $\dfrac{1}{c}\dfrac{\partial M_{r\theta}}{\partial\theta} = Q_r$ on C $(\varrho = 1)$.

4. Sectorial Plate Simply Supported along the Radial Edges, Free along the Circular Edge and Loaded over a Concentric Sector. The results obtained at the end of the previous section enable us to obtain the appropriate solution for a sectorial plate having a free circular edge and bounded by the two simply supported radii $\theta = 0$, $\theta = \beta = \pi/m$ (m an integer >1) when the plate is bent by the transverse pressure $p = p_0 r^{s-2}$ $(s \geq 2)$ distributed over the area of a concentric sector of radius $b \leq c$. It is clear that the deflexion of this plate is the same as that of the entire circular plate when the latter is subject to the load

$$\left.\begin{aligned} p &= p_0 r^{s-2} \text{ for } 2j\beta \leq \theta \leq 2(j+1)\beta, \\ p &= -p_0 r^{s-2} \text{ for } (2j+1)\beta \leq \theta \leq 2(j+1)\beta \;\; (j = 0, 1, 2, \ldots, m-1,) \end{aligned}\right\} \tag{4.1}$$

over region 1 defined by $0 \leq r \leq b$. This load can be represented by the Fourier series

$$p = \frac{4mp_0}{\pi} r^{s-2} {\sum}' \frac{\sin k\theta}{k}, \tag{4.2}$$

where the prime on Σ means that the summation is taken only over positive odd multiples of m. The deflexion of a circular plate having a free edge $r = c$ and subject to the load (4.2) over the area of a concentric circle of radius $b \leq c$ can be obtained by putting $\alpha = \frac{1}{2}\pi$, $\lambda = i$ in (3.22), (3.23), using (2.4), (2.2) and applying the principle of superposition. Introducing for convenience the notations

$$\varrho_1 = \varrho q,\ \varrho_2 = \varrho/q,\ \varrho = r/c,\ q = b/c,\ s' = s+2, \tag{4.3}$$

it is found that the deflexion at any point of the unloaded region is

$$w_2 = \frac{p_0 m b^{s'}}{2\pi D} {\sum}' \frac{\varrho_1{}^n}{\kappa} \biggl\{ \frac{n\varrho^2}{(n+1)(n+s')} - \frac{\varrho_2{}^2}{n+s} + \frac{(n^2+\kappa^2-1)q^{-2}}{n(n-1)(n+s)} - \frac{1}{n+s'} \biggr\}$$

$$+ \varrho_2{}^{-n} \biggl\{ \frac{1}{(n+1)(n+s')} - \frac{\varrho_2{}^2}{(n-1)(n+s)} \biggr\} \biggr] \frac{\sin n\theta}{n^2}. \tag{4.4}$$

For the deflexion at any point of the loaded region we have the three following cases:

Case (*a*). Neither s nor s' is an odd multiple of m.

$$w_1 = \frac{p_0 m b^{s'}}{2\pi D} {\sum}' \biggl[\frac{\varrho_1{}^n}{k} \biggl\{ \frac{n\varrho^2}{(n+1)(n+s')} - \frac{\varrho_2{}^2}{n+s} + \frac{(n^2+\kappa^2-1)q^{-2}}{n(n-1)(n+s)} - \frac{1}{n+s'} \biggr\}$$

$$+ \varrho_2{}^n \biggl\{ \frac{1}{(n-1)(n-s')} - \frac{\varrho_2{}^2}{(n+1)(n-s)} \biggr\} - \frac{8n\varrho_2{}^{s'}}{(n^2-s^2)(n^2-s'^2)} \biggr] \frac{\sin n\theta}{n^2}. \tag{4.5a}$$

Case (*b*). s is on odd multiple of m.

$$w_1 = \frac{p_0 m b^{s'}}{2\pi D} \biggl[{\sum}' \frac{\varrho_1{}^n}{\kappa n^2} \biggl\{ \frac{n\varrho^2}{(n+1)(n+s')} - \frac{\varrho_2{}^2}{n+s} + \frac{(n^2+\kappa^2-1)q^{-2}}{n(n-1)(n+s)} - \frac{1}{n+s'} \biggr] \sin n\theta$$

$$+ \sum_{n \text{ not equal to } s}{}' \frac{\varrho_2{}^n}{n^2} \biggl\{ \frac{1}{(n-1)(n-s')} - \frac{\varrho_2{}^2}{(n+1)(n-s)} - \frac{8n\varrho_2{}^{s'-n}}{(n^2-s^2)(n^2-s'^2)} \biggr\} \sin n\theta$$

$$+ \frac{\varrho_2{}^s}{2s^2} \biggl\{ \frac{\varrho_2{}^2}{s} + \frac{\varrho_2{}^2}{(s+1)^2} - \frac{1}{s-1} - \frac{2\varrho_2{}^2 \log \varrho_2}{s+1} \biggr\} \sin s\theta \biggr]. \tag{4.5b}$$

Case (*c*). s' is an odd multiple of m.

$$w_1 = \frac{p_0 m b^{s'}}{2\pi D} \biggl[{\sum}' \frac{\varrho_1{}^n}{\kappa n^2} \biggl\{ \frac{n\varrho^2}{(n+1)(n+s')} - \frac{\varrho_2{}^2}{n+s} + \frac{(n^2+\kappa^2-1)q^{-2}}{n(n-1)(n+s)} - \frac{1}{n+s'} \biggr\} \sin n\theta$$

$$+ \sum_{n \text{ not equal to } s'}{}' \frac{\varrho_2{}^n}{n^2} \biggl\{ \frac{1}{(n-1)(n-s')} - \frac{\varrho_2{}^2}{(n+1)(n-s)} - \frac{8n\varrho_2{}^{s'-n}}{(n^2-s^2)(n^2-s'^2)} \biggr\} \sin n\theta$$

$$+ \frac{\varrho_2{}^{s'}}{2s'^2} \biggl\{ \frac{1}{s'} - \frac{1}{(s'-1)^2} - \frac{\varrho_2{}^2}{s'+1} + \frac{2 \log \varrho_2}{s'-1} \biggr\} \sin s'\theta \biggr]. \tag{4.5c}$$

The maximum deflexion occurs at the mid-point of the free circular edge.

Setting $\varrho = 1$, $\theta = \pi/2m$ in (4.4) and substituting for κ form (2.8) we get

$$w_{\max} = \frac{2p_0 b^s}{\pi m(3+\nu)D} \sum_{1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{1}{2}(j-1)}}{j^3}\left[\frac{q^{jm}}{(jm+1)(jm+s')} + \frac{q^{jm-2}}{jm(jm+s)}\left(\frac{\kappa}{jm-1}-1\right)\right] \quad (4.6)$$

As a check on the validity of this formula we shall apply it to the special case of a uniformly loaded quadrant. Putting $m = 2$, $q = 1$, $s = 2$, $s' = 4$ and $\nu = 0.3$ in (4.6) and substituting for D from (2.5) we obtain

$$w_{\max} = 0.691 p_0 c^4/Eh^3,$$

in agreement with Timoshenko's result (1940, p. 274).

Setting $\alpha = \frac{1}{2}\pi$ in (3.24), using the Fourier series (4.2) and applying the principle of superposition we arrive at the following expressions for the moments and shears at any point (r, θ) of the unloaded region:

$$M_r = \frac{(1-\nu)mp_0 b^s}{2\pi}\sum{}' \frac{\sin n\theta}{n}\left[\frac{\varrho_1^{\,n}}{\kappa}\left\{(n-\kappa-1)\left(\frac{q^2}{n+s'} - \frac{n+1}{n(n+s)}\right)\right.\right.$$

$$\left.\left.- \frac{n-1}{n+s'}\varrho_2^{\,-2} + \frac{n^2+\kappa^2-1}{n(n+s)}\varrho^{-2}\right\} + \varrho_2^{\,-n}\left\{\frac{\varrho_2^{\,-2}}{n+s'} - \frac{n+\kappa+1}{n(n+s)}\right\}\right], \quad (4.7a)$$

$$M_\theta = \frac{(1-\nu)mp_0 b^s}{2\pi}\sum{}' \frac{\sin n\theta}{n}\left[\frac{\varrho_1^{\,n}}{\kappa}\left\{(n+\kappa+1)\left(\frac{n+1}{n(n+s)} - \frac{q^2}{n+s'}\right)\right.\right.$$

$$\left.\left.+ \frac{n-1}{n+s'}\varrho_2^{\,-2} - \frac{n^2+\kappa^2-1}{n(n+s)}\varrho^{-2}\right\} + \varrho_2^{\,-n}\left\{\frac{n-\kappa-1}{n(n+s)} - \frac{\varrho_2^{\,-2}}{n+s'}\right\}\right], \quad (4.7b)$$

$$M_{r\theta} = \frac{(1-\nu)mp_0 b^s}{2\pi}\sum{}' \frac{\cos n\theta}{n}\left[\frac{\varrho_1^{\,n}}{\kappa}\left\{\frac{n+1}{n+s} - \frac{nq^2}{n+s'} + \frac{n-1}{n+s'}\varrho_2^{\,-2}\right.\right.$$

$$\left.\left.- \frac{n^2+\kappa^2-1}{n(n+s)}\varrho^{-2}\right\} + \varrho_2^{\,-n}\left\{\frac{\varrho_2^{\,-2}}{n+s'} - \frac{1}{n+s}\right\}\right], \quad (4.7c)$$

$$Q_r = \frac{2p_0 mb^s}{\pi r}\sum{}' \frac{\sin n\theta}{n}\left[\frac{\varrho_1^{\,n}}{\kappa}\left(\frac{nq^2}{n+s'} - \frac{n+1}{n+s}\right) - \frac{\varrho_2^{\,-n}}{n+s}\right], \quad (4.7d)$$

$$Q_\theta = \frac{2p_0 mb^s}{\pi r}\sum{}' \frac{\cos n\theta}{n}\left[\frac{\varrho_1^{\,n}}{\kappa}\left(\frac{nq^2}{n+s'} - \frac{n+1}{n+s}\right) + \frac{\varrho_2^{\,-n}}{n+s}\right]. \quad (4.7e)$$

On $\varrho = 1$ it is easily verified that $M_r = 0$, $\dfrac{\partial M_{r\theta}}{c\partial\theta} = Q_r$, as expected. The value of the bending moment M_θ at the mid-point of the free circular edge is found by putting

$\varrho = 1$, $\varrho_1 = q$, $\varrho_2 = q^{-1}$ and $\theta = \pi/2m$ in (4.7*b*). After some simplification and using (2.8) we get

$$(M_\theta)_{\rho=1,\ \theta=\pi/2m} = -\frac{2(1+\nu)p_0 b^s}{\pi\kappa} \sum_{1,3,5,\ldots}^{\infty} \frac{q^{jm}(-1)^{\frac{1}{2}(j-1)}}{j} \left[\frac{1}{jm+s}\left(1+\frac{1-\kappa}{jm}\right) - \frac{q^2}{jm+s'}\right]. \tag{4.8}$$

For a uniformly loaded quadrant ($m = 2$, $q = 1$, $s = 2$, $s' = 4$) equation (4.8) yields

$$(M_\theta)_{\rho=1,\theta=\frac{1}{4}\pi} = -\frac{1+\nu}{2\pi\kappa} p_0 c^2 \left[1 - \frac{1+\kappa}{2}\log 2 + (1-\kappa)\left(S - \frac{1}{4}\pi\right)\right], \tag{4.9}$$

where†
$$S = \sum_{1,3,5\ldots} (-1)^{\frac{1}{2}(n-1)}/n^2 = 0.915965.$$

For $\nu = 0.3$, $\kappa = -33/7$ the expression (4.9) gives

$$(M_\theta)_{\rho=1,\theta=\frac{1}{4}\pi} = 0.1831\ p_0 c^2,$$

which agrees with the value mentioned by Timoshenko (1940) at the bottom of p. 274.

5. **Infinite Perforated Plate Subject to a General Line Loading along a Circle.**

We now consider the transverse flexure of an infinitely large thin elastic plate in the z plane with an outer free edge and an inner free circular boundary C, with centre O and radius c. We assume that the plate is in equilibrium under the load (3.1) distributed along the boundary Γ of a circle with centre $z_0 = c\zeta_0 = r_0 e^{i\theta_0}$ ($|\zeta_0| > 1$) and radius $b \leq r_0 - c$. [See Fig. 1, p. 109 of Bassali (1958*b*)]. Here also the four complex potentials are taken in the forms (3.9), where $\Phi_0(\sigma)$ and $\Psi_0(\sigma)$ satisfy the boundary condition defined by (3.11) and (3.12). Assuming that $z = c\zeta$ where ζ is any point outside γ, following the usual procedure of multiplying (3.11) and its conjugate by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma - \zeta}$ and integrating around γ we get, apart from constant terms

$$\Phi_0(\zeta) = -\frac{1}{2\pi i\kappa}\int_\gamma \frac{f(\sigma)}{\sigma-\zeta}\,d\sigma, \tag{5.1a}$$

$$\Psi_0(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\bar{f}(\sigma^{-1})}{\sigma-\zeta}\,d\sigma - \frac{1}{\zeta}\Phi_0'(\zeta). \tag{5.1b}$$

It must be noticed that the coefficients of ζ and ζ^{-1} in the expansions of $\Phi(\zeta)$ and $\Psi(\zeta)$, respectively, must be taken as real. This follows from the following remarks: (1) the terms $iA_1\zeta$ in $\Phi_0(\zeta)$ and $iA_2\zeta^{-1}$ in $\Psi_0(\zeta)$ (A_1 and A_2 real) do not affect the condition (3.11) for a free boundary, (2) the first term has no contribution to w while the second

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† See example 5, p. 526 of Bromwich (1942) for the sum of this series. A wrong value ($\pi^3/32$) was used by Timoshenko (1940), p. 172. It seems that Timoshenko confused this series with

$$\sum_{1,3,5,\ldots}^{\infty} (-1)^{\frac{1}{2}(n-1)}/n^3 = \pi^3/32$$ (ex 2, p. 364 of Bromwich, 1942).

term leads to the term $icA_2 \log \zeta$ in $\chi_0(\zeta)$ and therefore contributes a multiple-valued term to the deflexion which must be a single-valued function.

Substituting from (3.12) in (5.1), evaluating the integrals and leaving out the imaginary parts of the coefficients of ζ, ζ^{-1} in $\Phi_0(\zeta)$, $\Psi_0(\zeta)$, respectively we find

$$\Phi_0(\zeta) = \frac{\delta bc}{\kappa k}\left[\bar{\lambda}\left\{u_0 - \frac{kq^2\zeta}{(k+1)v_0}\right\}\left(\frac{g\zeta}{v_0}\right)^k - (-q)^k \zeta \,\mathbf{Re}\,(\lambda\zeta_0^{-k})\right] + C, \tag{5.2a}$$

$$\Psi_0(\zeta) = -\frac{\delta bc}{\kappa k\zeta}\left[\bar{\lambda}\left\{1 + \frac{\kappa^2 v_0}{k-1} + \frac{ku_0}{\zeta v_0} - \frac{kq^2}{v_0^2}\right\}\left(\frac{q\zeta}{v_0}\right)^k - (-q)^k (1+\kappa)\,\mathbf{Re}\,(\lambda\zeta_0^{-k})\right], \tag{5.2b}$$

where the same notations (3.15) are used. With these values introduced ia (3.9), the complete solution of the problem under consideration is given by

$$\varphi(z) = \frac{\delta b}{k}\left[\frac{c\bar{\lambda}}{\kappa}\left\{u_0 - \frac{kq^2\zeta}{(k+1)v_0}\right\}\left(\frac{q\zeta}{v_0}\right)^k - (-q)^k \frac{z}{\kappa}\,\mathbf{Re}\,(\lambda\zeta_0^{-k}) + \mathrm{F}(Z)\right] + C, \tag{5.3a}$$

$$\psi(z) = -\frac{\delta b}{k}\left[\frac{c\bar{\lambda}}{\kappa\zeta}\left\{1 + \frac{\kappa^2 v_0}{k-1} + \frac{ku_0}{\zeta v_0} - \frac{kq^2}{v_0^2}\right\}\left(\frac{q\zeta}{v_0}\right)^k - \frac{c(1+\kappa)}{\kappa\zeta}(-q)^k\,\mathbf{Re}\,(\lambda\zeta_0^{-k}) + G(Z)\right], \tag{5.3b}$$

where $F(Z)$ and $G(Z)$ are the same functions defined by (3.17) and (3.18). It can be verified that these functions satisfy the boundary conditions $M_r = 0$, $Q_r - \dfrac{\partial M_{r\theta}}{r\partial_\theta} = 0$ on $r = c$ and as $r \longrightarrow \infty$.

If was shown† (Bassali, 1957*c*, 1958*b*) that the functions φ, ψ which solve the problem of an infinitely large plate containing a free circular hole and subject to the normal system of forces N_s at the points z_s $(s = 1,2,\ldots,m)$ are

$$\varphi(z) = \frac{c}{16\pi\kappa D}\sum_{s=1}^{m} N_s u_s\left(\kappa \log u_s - \log \frac{v_s}{\zeta\bar{\zeta}_s}\right) + C', \tag{5.4a}$$

$$\psi(z) = -\frac{c}{16\pi\kappa D}\sum_{s=1}^{m} N_s\left[\bar{\zeta}_s\left(\kappa \log u_s - \log\frac{v_s}{\zeta\bar{\zeta}_s}\right) + \frac{1}{\zeta}\left\{\frac{1-\varrho_s^2}{v_s} + (\kappa^2-1)v_s \log\frac{v_s}{\zeta\bar{\zeta}_s} + \right.\right.$$
$$\left.\left. + \kappa(1+\kappa)\log \varrho_s\right\}\right], \tag{5.4b}$$

where the notations (3.20) are adopted. Adding (5.3*a*) to (5.4*a*) and (5.3*b*) to (5.4*b*) we obtain the solution appropriate to an infinite plate with a free circular hole when the plate is supported at any number of points and is subject to the load (3.1) along the boundary of a circle.

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† Only the real part of the coefficient of ζ^{-1} in the expansion of the function $\Psi_0(\zeta)$ given by equatron (3.29), p. 748 of Bassali (1957*c*) should be taken.

6. Half-plane Normally Loaded along the Circumference of a Circle. The solution (8.16) established for a circular plate with free edge and subject to the pressure (3.1) along Γ may be written as

$$\varphi(z) = \frac{\delta b}{k} F(Z) + \varphi_0(z), \quad \psi(z) = -\frac{\delta b}{k} G(Z) + \psi_0(z), \tag{6.1}$$

where

$$\varphi_0(z) = \frac{\delta b\bar{\lambda}}{k} \left\{Z - \frac{kb^2 z}{(k+1)(c^2 - \bar{z}_0 z)}\right\} \left(\frac{bz}{c^2 - \bar{z}_0 z}\right)^k + C, \tag{6.2a}$$

$$\psi_0(z) = -\frac{\delta b\bar{\lambda}}{k\kappa z} \cdot \left\{c^2 + \frac{\kappa^2(c^2 - \bar{z}_0 z)}{k-1} + \frac{c^4 kZ}{z(c^2 - \bar{z}_0 z)} - \frac{kb^2c^4}{(c^2 - \bar{z}_0 z)^2}\right\} \left(\frac{bz}{c^2 - \bar{z}_0 z}\right)^k \tag{6.2b}$$

We now consider how these functions change under change of origin from the centre O of the plate to the point $O'(z = -c)$. Assuming that z' denotes the complex variable of any point of the circular plate referred to O' as origin we have $z = z' - c$, $z_0 = z_0' - c$ so that $Z = z - z_0$, $F(Z)$ and $G(Z)$ remain unchanged and the new functions become

$$\varphi = \frac{\delta b}{k} F(Z) + \varphi_0(z' - c), \tag{6.3a}$$

$$\psi = -\frac{\delta b}{k} G(Z) + \psi_0(z' - c) - c\varphi_0'(z' - c), \tag{6.3b}$$

where in $\varphi_0(z' - c)$, $\varphi_0'(z' - c)$ and $\psi_0(z' - c)$ we put $z_0' - c$, $\bar{z}_0' - c$ instead of z_0, $\bar{z}_0$, respectively whenever they occur. Proceeding to the limit as c tends to infinity we get the following results for the half-plane with free edges when the plate is acted upon by the normal loading (8.1) along Γ:

$$\varphi(z) = \frac{\delta b}{k} \left[\frac{\bar{\lambda}}{\kappa} \left(Z + \frac{k}{k+1} \frac{b^2}{Z'}\right) \left(-\frac{b}{Z'}\right)^k + F(Z)\right] + C, \tag{6.4a}$$

$$\psi(z) = -\frac{\delta b}{k} \left[\frac{\bar{\lambda}}{\kappa} \left\{\frac{kz}{Z'} \left(Z + \frac{b^2}{Z'}\right) - \frac{\kappa^2 Z'}{k-1} - z\right\} \left(-\frac{b}{Z'}\right)^k + G(Z)\right], \tag{6.4b}$$

where $z = x + iy$ is now the complex variable of any point in the half-plane $x \geq 0$ referred to the new origin on the straight edge which is chosen as the y-axis, z_0 is the centre of Γ referred to the same origin, $Z = z - z_0$, $Z' = z + \bar{z}_0$ and the functions $F(Z)$, $G(Z)$ are given by (3.17), (3.18), respectively. [See Fig. 3, p. 280 of Bassali (1958*a*)]. It is verified that the same functions (6.4) are arrived at by changing the origin in the results of section 5 to the point $z = c$ and then letting c tend to ∞. The moments and shears at any point of the half-plane may be computed by applying the well-known formulae

$$\left.\begin{aligned} M_x + M_y &= -8D(1+\nu)\,\mathbf{Re}\,\varphi'(z), \\ M_x - M_y + 2iM_{xy} &= -4D(1-\nu)\,[\bar{z}\,\varphi''(z) + \psi'(z)], \\ Q_x - iQ_y &= -8D\,\varphi''(z). \end{aligned}\right\} \tag{6.5}$$

From (6.4) and (6.5) it is checked that $M_x = 0$ on $x = 0$.

The solution for a half-plane bent by the system of normal isolated forces N_s at the points z_s $(s = 1,2,\ldots,m)$ was obtained (Bassali, 1957*b*, p. 742) as

$$\varphi(z) = \frac{1}{16\pi D}\sum_{s=1}^{m} N_s Z_s \left(\log Z_s - \frac{1}{\kappa}\log Z_s'\right) + C'. \tag{6.6a}$$

$$\psi(z) = -\frac{1}{16\pi\kappa D}\sum_{s=1}^{m} N_s\left[\bar{z}_s\left(\kappa\log Z_s - \log Z_s' + \frac{2x_s}{Z_s'}\right) + (1-\kappa^2)\, Z_s'\log Z_s'\right]. \tag{6.6b}$$

Adding (6.4*a*) to (6.6*a*) and (6.4*b*) to (6.6*b*) we get the functions which solve the problem of a half-plane supported at any number of interior or boundary points and subject to the load (3.1) along Γ. Solutions corresponding to loadings of the type $p = p_0 R^m \cos(n\Theta - \alpha)\,(n > 1)$ distributed over the circular area bounded by Γ can be deduced by applying the principle of superposition.

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ON TOTAL INCLUSION FOR NÖRLUND METHODS OF SUMMABILITY

By

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Hardy (1949) has discussed ordinary inclusion for Nörlund matrices and Nörlund-type matrices. Debi (1955) and Lorch (1957) have extended some of his results to total inclusion. It is the purpose of this paper to continue the investigations started by Debi (1955) and Lorch (1957).

Let (N, p_n) denote a Nörlund method; *i.e.*, $\{p_n\}$ is a sequence of real numbers with $p_0 > 0$, $p_n \geq 0$, $P_n = \sum_{k=0}^{n} p_k$, with associated matrix $A = (a_{nk})$, where $a_{nk} = p_{n-k}/P_n$, $k \leqq n$, and $a_{nk} = 0$ for $k > n$.

Hardy, (1949, p. 57) has described a regular method of summation, denoted by $(\overline{N}, p_n)$, where p_n are as described above, and the associated matrix $B = (b_{nk})$ has the property that $b_{nk} = p_k/P_n$, $k \leqq n$, $b_{nk} = 0$ for $k > n$.

Theorem 1. *Let $\{p_n\}$ be a real positive sequence, with $P_n \to +\infty$. Then (N, p_n) t.s. $(\overline{N}, p_n)$ if $p_{n+1} \geq p_n$ for almost all n.*

For a definition of the concept of *totally stronger* (t.s.), see (Basu, 1948-9, p. 447).

Let $u_n = \sum_{k=0}^{n} a_{nk} s_k$, $v_n = \sum_{k=0}^{n} b_{nk} s_k$. Then

$$v_n = \sum_{k=0}^{n} p_k s_k / P_n = (P_{n-1} v_{n-1})/P_n + p_n s_n / P_n,$$

$$s_n = p_n^{-1} (P_n v_n - P_{n-1} v_{n-1}),\ P_{-1} = 0 \text{ and } u_n = \sum_{k=0}^{n} b_{nk} p_k^{-1} (P_k v_k - P_{k-1} v_{k-2})$$

$$= \sum_{k=0}^{n} P_k (b_{n,k}/p_k - b_{n,k+1}/p_{k+1}) v_k. \quad \text{Hence } A = DB, \text{ where}$$

$d_{nk} = P_k (b_{n,k}/p_k - b_{n,k+1}/p_{k+1})$ with the associated matrix $D = (d_{nk})$. Note that $\lim_{n \to \infty} d_{nk} = 0$,

$$\sum_{k=0}^{n} d_{nk} = \sum_{k=0}^{n} b_{nk} p_k^{-1} (P_k - P_{k-1}) = \sum_{k=0}^{n} b_{nk} p_k^{-1} (p_k) = \sum_{k=0}^{n} b_{nk} = 1,$$

and that d_{nk} may be written as $P_k(p_{n-k}/p_k - p_{n-k-1}/p_{k+1})/P_n$. For n and k sufficiently large, $p_{k+1} \geq p_k$, and $p_{n-k} \geq p_{n-k-1}$. Thus $p_k^{-1} \geq p_{k+1}^{-1}$, $d_{nk} \geq 0$ for almost all k, and D has finite norm. Therefore D is regular and also totally regular. (For conditions for regularity see [(Hardy, 1949, p. 43)].

If $p_{n+1} < p_n$ for a infinite number of values of n, then $d_{nk} < 0$ for an infinite number of values of n, and D cannot be totally regular by, Hurwitz (1927), Theorem 2 or (Hardy 1949, p. 53), Theorem 10.

Theorem 2. *If* $p_n > 0$, $q_n > 0$, $\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} q_k = \infty$, *then* $(\bar{N}, q_n)$ t.s. $(\bar{N}, p_n)$ *iff* $q_{n+1}/q_n \leq p_{n+1}/p_n$ *for almost all* n.

Theorem 2 is contained in Lorch (1957, p. 163), and thus a proof will be omitted.

Let Γ_β^α denote the Hausdorff summability method with $[\beta/(n+\beta)]^\alpha$ as generating sequence, $\alpha, \beta > 0$. Γ_β^α is defined in Hausdorff (1921) and used in Basu (1957).

Theorem 3. *Let* $\{p_k\}$ *be a sequence of positive numbers such that* $p_k \geq (k+1)p_{k+1}/(k+c)$, $c > 0$, *for almost all* k. *Then* $(\bar{N}, p_n)$ t.s. Γ_c^1, *but not conversely.*

Proof. If $H = (h_{nk})$ denotes the Hausdorff matrix Γ_c^1, then $h_{nk} = \left(\frac{n}{n+c}\right) h_{n-1,k}$.

Let s denote a sequence. Then

$$H_n(s) = \sum_{k=0}^{n} h_{nk} s_k = \sum_{k=0}^{n-1} h_{nk} s_k + \left(\frac{c}{n+c}\right) s_n$$

$$= \sum_{k=0}^{n-1} \left(\frac{n}{n+c}\right) h_{n-1,k} s_k + \left(\frac{c}{n+c}\right) s_n = \left(\frac{n}{n+c}\right) H_{n-1}(s) + \left(\frac{c}{n+c}\right) s_n.$$

Therefore $s_n = c^{-1}[(n+c)H_n(s) - nH_{n-1}(s)]$, $n = 0, 1, 2, \ldots$, $H_{-1}(s) = 0$.

Let $u_n = \sum_{k=0}^{n} e_{nk} s_k$. Then

$$u_n = \sum_{k=0}^{n} e_{nk} c^{-1}[(k+c)H_k(s) - kH_{k-1}(s)]$$

$$= \sum_{k=0}^{n} c^{-1}[(k+c)e_{nk} - (k+1)e_{n\,k+1}] H_k(s).$$

$E = MH$, where $m_{nk} = c^{-1}[(k+c)e_{nk} - (k+1)e_{n,k+1}]$. Note that $\lim_{n\to\infty} m_{nk} = 0$ and that $\sum_{k=0}^{n} m_{nk} = 1$. Also, m_{nk} can be written as

$$\frac{1}{c}\left\{\frac{k+c)p_k}{P_n} - \frac{(k+1)p_{k+1}}{P_n}\right\} = \frac{f(c,k)}{cP_n},$$

where $f(c, k) = (k+c)p_k - (k+1)p_{k+1}$.

By hypothesis $f(c,k) \geq 0$ for almost all k. Therefore $d_{nk} \geq 0$ for almost all k. Hence D has finite norm, is regular, and also totally regular by (Hardy, 1949 p. 53), Therem 10.

To show that the converse does not hold, note that, from $u_n = \sum_{k=0}^{n} e_{nk} s_k$, $s_n = p_n^{-1}(P_n u_n - P_{n-1} u_{n-1})$, $P_{-1} = u_{-1} = 0$. Then

$$H_n(s) = \sum_{k=0}^{n} h_{nk} p_k^{-1} (P_k u_k - P_{k-1} u_{k-1})$$

$$= \sum_{k=0}^{n} P_k \left(\frac{h_{n,k}}{p_k} - \frac{h_{n,k+1}}{p_{k+1}} \right) u_k.$$

Therefore $H = GE$, where

$$g_{nk} = P_k \left(\frac{h_{nk}}{p_k} - \frac{h_{n,k+1}}{p_{k+1}} \right) = \frac{c P_k n! \Gamma(k+c)}{p_k p_{k+1} (k+1)! \Gamma(n+c+1)} [-f(c,k)].$$

Note that $\lim_{n\to\infty} g_{nk} = 0$ and that $\sum_{k=0}^{n} g_{nk} = 1$.

Since $-f(c,k) < 0$ for almost all k, G has finite norm and is therefore regular but not totally regular.

Theorem 7 of [Debi, (1955), p. 140] states, "Both the methods, the $\left(\overline{N}, \frac{1}{n+1}\right)$ method and the logarithmic mean method, are stronger than $(C, 1)$ for infinite limits". This theorem is a special case of Theorem 3 with $p_n = (n+1)^{-1}$, $c = 1$, since $\Gamma_1^1 = (C, 1)$.

I quote the following two theorems from Debi (1955).

Theorem D2. (p. 136) *If* (N, q_n) *is a regular Nörlund method with* $q_n \uparrow$, *then* $s_n \to +\infty (C, 1)$ *implies* $s_n \to +\infty (N, q_n)$.

Theorem D4. (p. 137) $(N, \cosh n^{\frac{1}{2}})$ *does not totally include* (C, α) *for* $a \geqq 1.55$.

Theorem D2 for $s_n \to l$, l finite is a well-known result. Theorem D4 leaves as an open question what happens for $0 \leqq \alpha \leqq \cosh 1$.

A partial answer is

Theorem 4. $(N, \cosh n^{\frac{1}{2}})$ *t.s.* (C, α) *for* $0 \leqq \alpha \leqq 1$.

Proof. With $q_n = \cosh n^{\frac{1}{2}}$, $(N, \cosh n^{\frac{1}{2}})$ t.s. $(C, 1)$ from Theorem D2. Since C^1 t.s. C^α, $0 \leqq \alpha < 1$, $(N, \cosh n^{\frac{1}{2}})$ t.s. (C, α), $0 \leqq \alpha \leqq 1$.

I conjecture that Theorem 4 is true for $0 \leqq \alpha \leqq \cosh 1$.

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PROBLEMS OF DEEP-WATER WAVES. PART I. THE EXACT AND ASYMPTOTIC SOLUTIONS.

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ABSTRACT

The deep-water wave problem of Cauchy and Poisson is presented in a general three-dimensional setting to include in it all possible asymmetric initial surface disturbances of the Terazawa (1916) type. It has been possible to show that (*i*) the classical exact solutions are particular cases of the general expressions obtained here, (*ii*) the complete asymptotic solution beyond the first order term can be obtained for the case of finitely extended disturbances and that, in this case, the second and, presumably, the higher order terms are not given by the methods known before.

INTRODUCTION

The classical problem of deep-sea water waves generated by time-independent natural surface disturbances in three dimensions was discussed by Lamb (1932) for the case of symmetry of the wave motion about the origin, and later Terazawa (1916) extended his results to include certain types of asymmetry of the motion. The few exact solutions obtained by them were all derived for particular space distributions of the initial disturbances which again were either concentrated or extended over an infinite region but steadily decayed to zero at infinity. No exact solutions of the problem have, however, been given for arbitrary spatial distributions of the initial surface disturbances and, in particular, for the really interesting cases of the disturbances extended over a finite zone of the free surface. These solutions are obtained first in this paper.

The present-day discussions of the problem are mainly concerned, as these always have been since the days of Cauchy and Poisson, with finding an asymptotic representation of the motion by approximating methods of which the Stokes-Kelvin method of stationary phase has now been adopted by the modern writers such as Lamb (1932), Sneddon (1951) and Stoker (1957) as the basis of their discussions of this problem. All the current approximating methods have, however, the limitations: (α) in some cases, these are unable to give the terms of the second and higher orders of smallness, this remark applying, as shown here, specifically to the second order terms obtained by Hayasi (1939) on the method of stationary phase and by Wiedenbauer (1934) on the saddle-point method in their asymptotic solutions of the problem for the particular cases of an initial cylindrical surface impulse and a similar elevation respectively; (β) their applications to the general problem with arbitrary laws of surface disturbances remain to be justified in full. In this paper, a method is given for finding the complete asymptotic development of the wave integral for arbitrary laws of finitely extended disturbances. The first order term of this expansion has been found to agree with that deduced by the method of stationary phase. The approximate solutions in some interesting particular cases are next obtained.

PROBLEM

A heavy inviscid incompressible fluid, at rest at times $t<0$, fills the entire semi-infinite space $z\leq 0$, the xy-plane being taken in the undisturbed free surface and the z-axis being drawn vertically upwards. A wave motion is set up by the action of time-independent natural disturbances on the surface $z=0$ of the fluid at time $t=0$. If φ be the velocity potential of the irrotational motion so generated, we have the equation of continuity

$$\nabla^2\varphi = 0 \tag{1}$$

and Bernoulli's pressure equation

$$\frac{p}{\varrho}-\frac{\partial\varphi}{\partial t}+\tfrac{1}{2}q^2+gz = F(t)$$

where p denotes the pressure at (x, y, z) at time t, ϱ uniform density of the fluid, q the speed at (x, y, z) at time t, g the acceleration due to gravity and F an arbitrary function of t. This gives, on absorbing $F(t)$ in $\frac{\partial\varphi}{\partial t}$ and assuming small oscillations in which squares of the velocities are negligible,

$$\frac{p}{\varrho}=\frac{\partial\varphi}{\partial t}-gz. \tag{2}$$

If the surface is subject to a uniform atmospheric pressure p_0, the surface elevation ζ at the point $(x, y, 0)$ at time t above the undisturbed surface is, by (2),

$$\zeta = \frac{1}{g}\left(\frac{\partial\varphi}{\partial t}\right)_{z=0}, \tag{3}$$

on absorbing a constant p_0/ϱ in $\frac{\partial\varphi}{\partial t}$.

The kinematical surface condition gives, to the above degree of approximation,

$$\frac{\partial\zeta}{\partial t}=-\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} \tag{4}$$

Hence

$$\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial z}=0 \qquad \text{on } z=0. \tag{5}$$

If the time-factor be $e^{i(\sigma t+\epsilon)}$, equation (5) gives

$$\sigma^2\varphi = g\frac{\partial\varphi}{\partial z} \qquad \text{on } z=0. \tag{6}$$

The condition at infinity is

$$\varphi \to 0 \qquad \text{as } z \to -\infty. \tag{7}$$

For the initial values of the surface disturbances, we assume

$$(\varrho\varphi)_{\substack{z=0\\t=0}} = F_1(r)\cos m\theta + F_2(r)\sin m\theta, \tag{8A}$$

$$(\zeta)_{\substack{z=0\\t=0}} = f_1(r)\cos m\theta + f_2(r)\sin m\theta, \tag{8B}$$

where (r, θ, z) are cylindrical co-ordinates and m is zero or a positive integer. The problem is to find a solution $\varphi(r, \theta, z, t)$ which satisfies the equations (1), (6), (7) and (8A) and (8B) and ζ is then obtained by (3).

SOLUTION

A typical solution of (1), which is finite at origin and which satisfies the equations (6) and (7), is

$$\varphi = J_m(kr)e^{kz}\left[\cos\sigma t(A_1\cos m\theta + A_2\sin m\theta) + \sin\sigma t(B_1\cos m\theta + B_2\sin m\theta)\right] \tag{9}$$

where

$$\sigma^2 = gk \tag{10}$$

and $\qquad A_s \equiv A_s(k, m),\ B_s \equiv B_s(k, m),\ s = 1, 2$ are independent of r, θ, z.

In our case, the wave number k varies from 0 to ∞, and the actual motion is the result of the superposition of the waves of all possible wave-lengths $2\pi/k$. Therefore, generally,

$$\varphi = \int_0^\infty J_m(kr)e^{kz}[\cos\sigma t(A_1\cos m\theta + A_2\sin m\theta) + \sin\sigma t\,(B_1\cos m\theta + B_2\sin m\theta)]dk \tag{11}$$

and from (3),

$$\zeta = \frac{1}{g}\int_0^\infty \sigma J_m(kr)[-\sin\sigma t\,(A_1\cos m\theta + A_2\sin m\theta) + \cos\sigma t(B_1\cos m\theta + B_2\sin m\theta)]dk \tag{12}$$

Introducing (8A) and (8B) in (11) and (12), we have

$$F_s(\alpha) = \varrho\int_0^\infty J_m(k\alpha)A_s dk, \tag{13}$$

$$f_s(\alpha) = \frac{1}{g}\int_0^\infty \sigma J_m(k\alpha)B_s dk \tag{14}$$

$$s = 1, 2.$$

Assuming that the functions $F_s(\alpha)$ and $f_s(\alpha)$ possess Hankel transforms of order m, we obtain (Titchmarsh, 1937)

$$A_s(k, m) = \frac{k}{\varrho}\int_0^\infty \alpha F_s(\alpha)J_m(k\alpha)d\alpha \tag{15}$$

$$B_s(k, m) = \frac{gk}{\sigma}\int_0^\infty \sigma f_s(\alpha)J_m(k\sigma)d\alpha \tag{16}$$

$$s = 1, 2.$$

Therefore, the general solutions of φ and ζ in the present problem are given by

$$\varphi = \varphi_1 \cos m\theta + \varphi_2 \sin m\theta \tag{17}$$

$$\zeta = \zeta_1 \cos m\theta + \zeta_2 \sin m\theta \tag{18}$$

where

$$\varphi_s = \frac{1}{\varrho} \int_0^\infty \left(\bar{F}_s \cos \sigma t + \frac{g\varrho}{\sigma} \bar{f}_s \sin \sigma t\right) k J_m(kr) e^{kz} dk \tag{19}$$

$$\zeta_s = \int_0^\infty \left(-\frac{\sigma}{g\varrho} \bar{F}_s \sin \sigma t + \bar{f}_s \cos \sigma t\right) k J_m(kr) dk, \tag{20}$$

$$s = 1, 2,$$

σ is given by (10), and $\bar{F}_s \equiv \bar{F}_s(k, m)$ and $\bar{f}_s \equiv \bar{f}_s(k, m)$ are the Hankel transforms of order m of the functions $F_s(\alpha)$ and $f_s(\alpha)$, given by the integrals on the right-hand sides of (15) and (16) respectively. With the sufficient restrictions (Titchmarsh, 1937) on $F_s(\alpha)$ and $f_s(\alpha)$ which validate the inversions as performed in (15) and (16), the integrals in (19) converge uniformly for $z \leq 0$ $t \geq 0$ and $r > r_0 > 0$. If $\bar{F}_s(k) = 0(k^{-1-n})$, $n > 0$, as $k \to \infty$ then $\partial/\partial t$ of these integrals can be taken under the integral sign and the integrals in (20) converge uniformly for $t \geq 0$ and $r > r_0 > 0$. In these cases, as also in some others (e.g. the classical example of concentrated disturbances) where the restrictions referred to above cannot be met but the results of (15) and (16) remain true, it is possible to obtain the surface values by retaining e^{kz} in the φ- and ζ-integrals and passing to the limit $z \to 0-$. We may then take

$$\zeta_s = \lim_{z \to 0-} \int_0^\infty \left(-\frac{\sigma}{g\varrho} \bar{F}_s \sin \sigma t + \bar{f}_s \cos \sigma t\right) k J_m(kr) e^{kz} dk. \tag{20A}$$

This limiting process is easily justified for concentrated disturbances from a study of the corresponding integrals (Stoker, 1957) or of the exact solutions as shown later on for disturbances on a finite area of the free surface.

The expressions (17)—(20A) include, as their particular cases, all the general integral solutions of the problem obtained before. For $m = 0$, $F_1(r)$ or $f_1(r) = 0$, the expressions reduce to Lamb's solutions [Lamb, 1932] for symmetrical wave motion about the origin. For $F_2(r) = 0$, $f_2(r) = 0$, $F_1(r)$ or $f_1(r) = 0$, the expressions agree with Terazawa's solutions [Terazawa, 1916, equations (12), (13), (66) and (67)].

EXACT EVALUATION OF THE VELOCITY POTENTIAL AND THE WAVE INTEGRALS.

Expanding $\cos \sigma t$ and $\sin \sigma t$ in powers of σt, and interchanging the orders of summation and k integration, which is clearly permissible, we get, from (19),

$$\varphi_s = \frac{1}{\varrho} \sum_{n=0}^{\infty} (-1)^n \frac{(gt^2)^n}{(2n)!} \int_0^\infty k^{n+1} \bar{F}_s J_m(kr) e^{kz} dk + \sum_{n=0}^{\infty} (-1)^n \frac{g^{n+1} t^{2n+1}}{(2n+1)!} \int_0^\infty k^{n+1} \bar{f}_s J_m(kr) e^{kz} dk \tag{21}$$

Introducing Hansen's integral for Bessel's Coefficients (Jahnke-Emde, 1945, viii (8*a*))

$$J_m(kr) = \frac{e^{im\pi/2}}{\pi} \int_0^{\pi} e^{-ikr\cos\psi} \cos m\psi d\psi$$

and interchanging the orders of k- and ψ-integrations, we have

$$\int_0^{\infty} k^{n+1}\bar{F}_s J_m(kr) e^{kz} dk = \frac{e^{im\pi/2}}{\pi} \int_0^{\pi} \cos m\psi d\psi \, \frac{\partial^n}{\partial z^n} \int_0^{\infty} k\bar{F}_s(k) e^{-k(ir\cos\psi - z)} dk. \tag{22}$$

Assuming that (*i*) the functions $k\bar{F}_s(k)$ have derivatives of all orders in $0 < k < \infty$, (*ii*) $\lim_{k\to 0+} [k\bar{F}_s(k)]^{(j)}$ (*i.e.* the j^{th} derivative of $k\bar{F}_s(k)$ *w.r.t.k*) exist for all j and (*iii*) the j-series is convergent for $r > r_0 > 0$, we get, by successive integration by parts,

$$\int_0^{\infty} k\bar{F}_s e^{-k(ir\cos\psi - z)} dk = \sum_{j=0}^{\infty} \frac{[k\bar{F}_s(k)]_{k=0+}^{(j)}}{(ir\cos\psi - z)^{j+1}} \tag{23}$$

and hence, from (22),

$$\int_0^{\infty} k^{n+1}\bar{F}_s J_m(kr) e^{kz} dk = \frac{e^{im\pi/2}}{\pi} \sum_{j=0}^{\infty} \frac{[k\bar{F}_s(k)]_{k=0+}^{(j)}}{j!} (j+n)! \int_0^{\pi} \frac{\cos m\psi d\psi}{(ir\cos\psi - z)^{j+n+1}}. \tag{24}$$

Using some properties and an integral representation of the associated Legendre functions $P_\nu^\mu(x)$ on the real axis in (0, 1) together with a Gamma-function result [Erdélyi, 1953, 3·4(7, 17), 3·7(14), 1·2(4)], we find

$$\int_0^{\pi} \frac{\cos m\psi d\psi}{(ir\cos\psi - z)^{j+n+1}} = \pi e^{im\pi/2} \frac{\Gamma(n+j-m+1)}{\Gamma(n+j+1)} \cdot \frac{P_{n+j}^{m}(z')}{(z^2+r^2)^{(n+j+1)/2}}, \; n+j \geq m = 0, 1, 2, \ldots$$

$$= \pi e^{-im\pi/2} \frac{\Gamma(n+j+m+1)}{\Gamma(n+j+1)} \cdot \frac{P_{n+j}^{-m}(z')}{(z^2+r^2)^{(n+j+1)/2}}, \; n+j < m = 1, 2, 3, \ldots$$

where

$$z' = -\frac{z}{(z^2+r^2)^{\frac{1}{2}}}. \tag{25}$$

Hence, from (24),

$$\int_0^{\infty} k^{n+1}\bar{F}_s J_m(kr) e^{kz} dk = \sum_{j=0}^{n+j=m-1} \frac{\Gamma(n+j+m+1)}{\Gamma(j+1)} [k\bar{F}_s(k)]_{k=0+}^{(j)} \frac{P_{n+j}^{-m}(z')}{(z^2+r^2)^{(n+j+1)/2}}$$

$$+ \sum_{\substack{j \\ n+j=m}}^{\infty} (-1)^m \frac{\Gamma(n+j-m+1)}{\Gamma(j+1)} [k\bar{F}_s(k)]_{k=0+}^{(j)} \frac{P_{n+j}^{m}(z')}{(z^2+r^2)^{(n+j+1)/2}}, \; m > 0 \tag{26A}$$

$$= \sum_{j=0}^{\infty} \frac{\Gamma(n+j+1)}{\Gamma(j+1)} [k\bar{F}_s(k)]_{k=0+}^{(j)} \frac{P_{n+j}(z')}{(z^2+r^2)^{(n+j+1)/2}}, \quad m = 0. \tag{26B}$$

The corresponding values of $\int_0^{\infty} k^{n+1}\bar{f}_s J_m(kr) e^{kz} dk$.

are obtained, under similar restrictions, on replacing $\bar{F}_s$ by $\bar{f}_s$ in (26*A* & *B*).

Using these, we get, from (21)

$$\varphi_s = \frac{(\pm 1)^m}{\varrho} \sum_n \sum_j^{(\Delta)} (-1)^n \frac{\Gamma(n+j\pm m+1)}{\Gamma(2n+2)\Gamma(j+1)} \cdot \frac{(gt^2)^n}{(z^2+r^2)^{(n+j+1)/2}} \cdot \overset{\mp m}{P_{n+j}}(z') \times$$

$$\{(2n+1)[k\bar{F}_s(k)]^{(j)}_{k=0+} + \varrho g t[k\bar{f}_s(k)]^{(j)}_{k=0+}\}, \tag{27}$$

where $\overset{(\Delta)}{\sum_n \sum_j}$ means: (*i*) for $m \geq 1$, the sum of the series $\sum_{n=0}^{n+j=m-1} \sum_{j=0}$ and $\sum_{\substack{n \ j \\ n+j=m}}^{\infty\ \infty} \sum$

(*ii*) for $m = 0$, the series $\sum_{n=0}^{\infty} \sum_{j=0}^{\infty}$,

and the ambiguity of sign in the series in case (*i*) is removed by taking the $\frac{\text{upper}}{\text{lower}}$ signs in the $\frac{\text{first}}{\text{second}}$ of these two series.

The double series obtained above are absolutely convergent and uniformly so for all r, z, t excluding a neighbourhood of the point ($r = 0$, $z = 0$) which is the centre of the disturbances.

By (3), we have now

$$\zeta_s = \frac{(\pm 1)^m}{g\varrho t} \sum_n \sum_j^{(\Delta')} (-1)^n \frac{\Gamma(n+j\pm m+1)}{\Gamma(2n)\Gamma(j-1)} \left(\frac{gt^2}{r}\right)^n [k\bar{F}_s(k\,]^{(j)}_{k=0+} \ \frac{\overset{\mp m}{P_{n+j}}(0)}{r^{j+1}}$$

$$+ (\pm 1)^m \sum_n \sum_j^{(\Delta)} (-1)^n \frac{\Gamma(n+j\pm m+1)}{\Gamma(2n+1)\Gamma(j+1)} \left(\frac{gt^2}{r}\right)^n [k\bar{f}_s(k)]^{(j)}_{k=0+} \ \frac{\overset{\mp m}{P_{n+j}}(0)}{r^{j+1}}, \tag{28}$$

where $\overset{(\Delta')}{\sum_n \sum_j}$ means: (*i*) for $m \geq 2$, the sum of the series $\sum_{n=1}^{n+j=m-1} \sum_{j=0}$ and $\sum_{\substack{n \ j \\ n+j=m}}^{\infty\ \infty} \sum$

(*ii*) for $m = 0, 1$, the series $\sum_{n=1}^{\infty} \sum_{j=0}^{\infty}$,

the actual sign in the series in case (*i*) being decided in the same way as before.

For the case of symmetry about the origin, $m = 0$. In this case, the solutions are

$$\varphi = (\varphi_1)_{m=0} = \frac{1}{\varrho} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^n \frac{\Gamma(n+j+1)}{\Gamma(2n+2)\Gamma(j+1)} \ \frac{(gt^2)^n}{(z^2+r^2)^{(n+j+1)/2}} P_{n+j}(z') \times$$

$$\{(2n+1)[k\bar{F}_1(k)\,]^{(j)}_{k=0+} + \varrho g t[k\bar{f}_1(k)]^{(j)}_{k=0+}\}. \tag{29}$$

$$\zeta = (\zeta_1)_{m=0} = \frac{1}{g\varrho t}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(-1)^n \frac{\Gamma(n+j+1)}{\Gamma(j+1)\Gamma(2n)}\left(\frac{gt^2}{r}\right)^n [k\bar{F}_1(k)]^{(j)}_{k=0+}\cdot\frac{P_{n+j}(0)}{r^{j+1}}$$

$$+\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^n \frac{\Gamma(n+j+1)}{\Gamma(j+1)\Gamma(2n+1)}\left(\frac{gt^2}{r}\right)^n [k\bar{f}_1(k)]^{(j)}_{k=0+}\cdot\frac{P_{n+j}(0)}{r^{j+1}}. \tag{30}$$

For the case $m = 1$, we have

$$(\varphi_s)_{m=1} = \frac{1}{\varrho}\,\frac{\overset{-1}{P_0}(z')}{(z^2+r^2)^{\frac{1}{2}}}\,\{[k\bar{F}_s(k)]_{k=0+} + \varrho gt[k\bar{f}_s(k)]_{k=0+}\}$$

$$-\frac{1}{\varrho}\sum_{\substack{n\\ n+j=1}}^{\infty}\sum_{j}^{\infty}(-1)^n \frac{\Gamma(n+j)}{\Gamma(2n+2)\Gamma(j+1)}\cdot\frac{(gt^2)^n}{(z^2+r^2)^{(n+j+1)/2}}\overset{1}{P}_{n+j}(z') \times$$

$$\{(2n+1)[k\bar{F}_s(k)]^{(j)}_{k=0+} + \varrho gt[k\bar{f}_s(k)]^{(j)}_{k=0+}\} \tag{31}$$

$$(\zeta_s)_{m=1} = \frac{P_0^{-1}(0)}{r}[k\bar{f}_s(k)]_{k=0+} - \sum_{j=1}^{\infty}\frac{1}{j}[k\bar{f}_s(k)]^{(j)}_{k=0+}\frac{\overset{1}{P}_j(0)}{r^{j+1}}$$

$$-\frac{1}{g\varrho t}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(-1)^n \frac{\Gamma(n+j)}{\Gamma(2n+1)\Gamma(j+1)}\left(\frac{gt^2}{r}\right)^n \frac{\overset{1}{P}_{n+j}(0)}{r^{j+1}} \times$$

$$\left\{2n[k\bar{F}_s(k)]^{(j)}_{k=0+} + \varrho gt[k\bar{f}_s(k)]^{(j)}_{k=0+}\right\}. \tag{32}$$

SPECIAL CONFIGURATIONS OF THE INITIAL SURFACE DISTURBANCES

The general expressions (27) and (28) include, as their particular cases, all the exact solutions of the problem obtained so far for different spatial distributions of the initial surface disturbances. Some of the classical solutions will now be obtained from (27) and (28) and the exact solutions, unknown hitherto, of the problem of finitely extended symmetrical disturbances will be deduced next in some simple cases of interest.

Concentrated disturbances : Lamb's solutions. (*a*) *Initial elevation without initial velocity.* Let $m = 0$, $F_1(r) = 0$, $f_1(r) = \dfrac{V}{2\pi r}\,\delta(r)$,

where V = total volume of the fluid displaced and $\delta(r)$ is Dirac's delta function.

$$\therefore\quad \bar{f}_1(k) = \frac{V}{2\pi}$$

and

$$[k\bar{f}_1(k)]^{(j)}_{k=0+} = \frac{V}{2\pi}, \quad j = 1$$

$$= 0, \qquad j \neq 1.$$

∴ Equations (29) and (30) give

$$\varphi = (\varphi_1)_{m=0} = \frac{Vgt}{2\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2)}{\Gamma(2n+2)} \frac{(gt^2)^n}{(z^2+r^2)^{(n+2)/2}} P_{n+1}\left(-\frac{z}{(z^2+r^2)^{\frac{1}{2}}}\right) \tag{33}$$

$$\zeta = (\zeta_1)_{m=0} = \frac{V}{2\pi r^2} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2)}{\Gamma(2n+1)} \left(\frac{gt^2}{r}\right)^n P_{n+1}(0)$$

$$\because \quad P_{2n+1}(0) = 0,\ P_{2n+2}(0) = (-1)^{n+1} \frac{1.3.5\ .\ (2n+1)}{2.4.6\ \ (2n+2)}, \tag{34}$$

we have

$$\zeta = \frac{V}{2\pi r^2}\left[\frac{1}{2!}.\ \frac{gt^2}{r} - \frac{1^2.3^2}{6!}\left(\frac{gt^2}{r}\right)^3 + \frac{1^2.3^2\ 5^2}{10!}\left(\frac{gt^2}{r}\right)^5 - \ldots\right] \tag{35}$$

(β) *Initial impulse without initial elevation,*

Let $m = 0,\ f_1(r) = 0,\ F_1(r) = \frac{P}{2\pi r}\delta(r),$

where P = total initial impulse upon the surface and $\delta(r)$ is, as before, Dirac's delta function.

We have then from (29) and (30)

$$\varphi = (\varphi_1)_{m=0} = \frac{P}{2\pi\varrho} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+2)}{\Gamma(2n+1)} \frac{(gt^2)^n}{(z^2+r^2)^{(n+2)/2}} P_{n+1}\left(-\frac{z}{(z^2+r^2)^{\frac{1}{2}}}\right) \tag{36}$$

$$\zeta = \frac{Pt}{2\pi\varrho r^3}\left[1 - \frac{1^2.3^2}{5!}\left(\frac{gt^2}{r}\right)^2 + \frac{1^2.3^2.5^2}{9!}\left(\frac{gt^2}{r}\right)^4 - \ldots\right] \tag{37}$$

The equations (33), (35), (36) and (37) are the well-known solutions obtained by Lamb [1932, 255(18, 20, 34)] for concentrated initial surface disturbances.

Extended disturbances decaying steadily to zero at infinity : Terazawa's solutions. Terazawa (1916) examined the cases

(i) $f_1(r) = \frac{A}{r}\ ;\ f_2(r) = F_1(r) = F_2(r) = 0,$

$= \frac{A}{(b^2+r^2)^{\frac{1}{2}}}, \frac{Ab}{(b^2+r^2)^{3/2}};\ F_1(r) = 0 \qquad \text{for } m = 0,$

(ii) $F_1(r) = \frac{A}{r},\ f_1(r) = f_2(r) = F_2(r) = 0,$

$= \frac{A}{(b^2+r^2)^{\frac{1}{2}}}, \frac{Ab}{(b^2+r^2)^{3/2}};\ f_1(r) = 0 \qquad \text{for } m = 0,$

all of which extend over the whole free surface though they decay steadily to zero at infinity. The solutions for all these cases may be easily deduced from the equations

(27) and (28). We deduce here only the solution for the first law of disturbances in (*i*). Here

$$f_1(k) = \frac{A}{k}$$

$$\therefore \quad [kf_1(k)]^{(j)}_{k=0+} = A, \quad j = 0$$
$$= 0, \quad j \neq 0.$$

Therefore, the equations (17) and (18) give, by (27) and (28),

$$\left.\begin{aligned} \varphi &= \varphi_1 \cos m\theta \\ \zeta &= \zeta_1 \cos m\theta \end{aligned}\right\} \tag{38}$$

where $$\varphi_1 = A\Bigg[\sum_{n=0}^{m-1} (-1)^n \frac{\Gamma(n+m+1)}{\Gamma(2n+2)} g^{n+1} t^{2n+1} \frac{P_n^{-m}\left(-\dfrac{z}{(z^2+r^2)^{\frac{1}{2}}}\right)}{(z^2+r^2)^{(n+1)/2}}$$

$$+(-1)^m \sum_{n=m}^{\infty} (-1)^n \frac{\Gamma(n-m+1)}{\Gamma(2n+2)} g^{n+1} t^{2n+1} \frac{P_n^m\left(-\dfrac{z}{(z^2+r^2)^{\frac{1}{2}}}\right)}{(z^2+r^2)^{(n+1)/2}}\Bigg], \tag{39}$$

$$\zeta_1 = \frac{A}{r}\Bigg[\sum_{n=0}^{m-1} (-1)^n \frac{\Gamma(n+m+1)}{\Gamma(2n+1)} \left(\frac{gt^2}{r}\right)^n P_n^{-m}(0) +$$

$$(-1)^m \sum_{n=m}^{\infty} (-1)^n \frac{\Gamma(n-m+1)}{\Gamma(2n+1)} \left(\frac{gt^2}{r}\right)^n P_n^m(0)\Bigg] \tag{40}$$

which agree with Terazawa's results (1916, equations 24, 25) if we note that the definition of $P_n^m(x)$, for $m > 0$, used by Terazawa differ by a factor $(-1)^m$ from that used here [Erdélyi, 1953, 3.6 (6, 10)] and also that z-axis is drawn there downwards.

Initial disturbances on a finite zone of the free surface. (*a*) *Paraboloidal Impulse.*

$$\left.\begin{aligned} \text{Let } m = 0, \quad f_1(r) = 0, \quad F_1(r) &= \frac{2P}{\pi a^4}(a^2 - r^2), \quad a \geq r \\ &= 0, \qquad\qquad\qquad a < r \end{aligned}\right\} \tag{41}$$

so that P is the total impulse on the surface.

$$\therefore \quad F_1(k) = \frac{4P}{\pi a^2} \cdot \frac{J_2(ka)}{k}.$$

$$\text{And} \quad [kF_1(k)]^{(j)}_{k=0+} = 0, \qquad j = 2l$$

$$= \frac{(-1)^l \, 4P \, a^{2l} \, \Gamma(2l+2)}{\pi 2^{2l+2} \, \Gamma(l+3) \, \Gamma(l+1)}, \qquad = 2l+1$$

where l = zero or any positive integer.

Therefore, from (29) and (30), the velocity potential and the surface elevation corresponding to the disturbances (41) are

$$\varphi = \frac{4P}{\pi\varrho}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{n+j}\frac{\Gamma(2j+n+2)}{2^{2j+2}\Gamma(2n+1)\Gamma(j+3)\Gamma(j+1)}\times$$

$$\cdot\ \frac{a^{2j}(gt^2)^n}{(z^2+r^2)^{(2j+n+2)/2}}P_{2j+n+1}\left(-\frac{z}{(z^2+r^2)^{\frac{1}{2}}}\right) \tag{42}$$

$$\zeta = \frac{4P}{\pi g\varrho tr^3}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}(-1)^{n+j}\frac{\Gamma(2j+n+2)}{2^{2j+2}\Gamma(2n)\Gamma(j+3)\Gamma(j+1)}\left(\frac{gt^2}{r}\right)^{n}\left(\frac{a}{r}\right)^{2j}P_{2j+n+1}(0),$$

both valid for $r > a$ at $t = 0$. From the last one, using (34),

$$\zeta = \frac{4P}{\pi g\varrho tr^2}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{n}\frac{\{\Gamma(2n+2j+3)\}^2}{2^{2n+4j+4}\Gamma(4n+2)\Gamma(j+3)\Gamma(j+1)\{\Gamma(j+n+2)\}^2}\times$$

$$\left(\frac{gt^2}{r}\right)^{2n+1}\left(\frac{a}{r}\right)^{2j}.$$

Using Legendre's duplication formula of Gamma functions [Erdélyi, 1953, 1.2(15)]

we get $$\{\Gamma(2n+2j+3)\}^2 = \frac{2^{4n+4j+4}}{\pi}\{\Gamma(n+i+\tfrac{3}{2})\}^2\ \{\Gamma(n+j+2)\}^2,$$

and so $$\zeta = \frac{2P}{\pi' g\varrho tr^2}\sum_{n=0}^{\infty}\sum_{j=0}(-1)^{n}\frac{\{\Gamma(n+j+\frac{3}{2})\}^2}{\Gamma(4n+2)\Gamma(j+3)\Gamma(j+1)}\left(\frac{2gt^2}{r}\right)^{2n+1}\left(\frac{a^2}{r^2}\right)^{j}. \tag{43A}$$

$$= \frac{P}{\pi^2 g\varrho tr^2}\sum_{n=0}^{\infty}(-1)^{n}\frac{\{\Gamma(n+\frac{3}{2})\}^2}{\Gamma(4n+2)}\left(\frac{2gt^2}{r}\right)^{2n+1}F\left(n+\tfrac{3}{2},\ n+\tfrac{3}{2},\ 3,\ \frac{a^2}{r^2}\right),\ r > a, \tag{43B}$$

where F denotes the hypergeometric series.

By using a relation between contiguous hypergeometric series (Ryshik-Gradstein 1957, 7·237(15)) and the expression for torus function $D^{m}_{n-\frac{1}{2}}(\cosh\eta)$ in terms of the hypergeometric series (Ryshik-Gradstein 1957, 6·872(1)), we have

$$\zeta = \frac{8Pt}{\pi^{5/2}\varrho a^{5/2}r^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{(n-\frac{1}{2})\Gamma(4n+2)}\left(\frac{2gt^2}{(r^2-a^2)^{\frac{1}{2}}}\right)^{2n}\left[(n+\tfrac{1}{2})D^{n}_{-\frac{1}{2}}\left(\frac{r^2+a^2}{2ar}\right)-\frac{r}{a}D^{n}_{\frac{1}{2}}\left(\frac{r^2+a^2}{2ar}\right)\right],$$

$$r > a. \tag{43C}$$

(β) *Ellipsoidal elevation and impulse.* (*i*) *Impulse.*

$$\left.\begin{aligned} \text{Let}\quad m = 0,\quad f_1(r) = 0,\quad F_1(r) &= \frac{3P}{2\pi a^3}(a^2-r^2)^{\frac{1}{2}}, & a \geq r\\ &= 0 &, \quad a < r \end{aligned}\right\} \tag{44}$$

so that P is the total initial impulse on the surface.

$$\therefore \qquad \bar{F}_1(k) = \frac{3P}{2\pi a^3 k^3}(\sin ka - ka \cos ka).$$

And
$$[k\bar{F}_1(k)]^{(j)}_{k=0+} = (-1)^l \, \frac{3P}{2\pi} \cdot \frac{a^{2l}}{2l+3}, \qquad j = 2l+1$$
$$= 0 \qquad , \qquad j = 2l$$

where $l =$ zero or any positive integer.

It follows from (29) and (30) that the velocity potential and the surface elevation corresponding to the disturbances (44) are

$$\varphi = \frac{3P}{2\pi\varrho} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+j} \frac{\Gamma(n+2j+2)}{\Gamma(2j+2) \; (2j+3)\Gamma(2n+1)} \cdot \frac{(gt^2)^n a^{2j}}{(z^2+r^2)^{(n+2j+2)/2}} P_{n+2j+1}\left(-\frac{z}{(z^2+r^2)^{\frac{1}{2}}}\right) \tag{45}$$

$$\zeta = \frac{3P}{2\pi\varrho\, gtr^2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+j} \frac{\Gamma(n+2j+2)}{\Gamma(2j+2) \; (2j+3)\Gamma(2n)} \left(\frac{gt^2}{r}\right)^n \left(\frac{a}{r}\right)^{2j} P_{n+2j+1}(0),$$

both of which are valid for $r > a$ at $t = 0$. Using (34) and Legendre's duplication formula for Gamma functions, we find

$$\zeta = \frac{3P}{4\pi^{3/2}\varrho\, gtr^2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^n \frac{\{\Gamma(n+j+\frac{3}{2})\}^2}{\Gamma(4n+2)\Gamma(j+\frac{5}{2})\Gamma(j+1)} \left(\frac{2gt^2}{r}\right)^{2n+1} \left(\frac{a^2}{r^2}\right)^j, \tag{46A}$$

$$= \frac{P}{\pi^2 g\varrho tr^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2gt^2}{r}\right)^{2n+1} \frac{\{\Gamma(n+\frac{3}{2})\}^2}{\Gamma(4n+2)} \; F\left(n+\tfrac{3}{2},\, n+\tfrac{3}{2},\, \tfrac{5}{2},\, \frac{a^2}{r^2}\right), \; (r > a). \tag{46B}$$

(*ii*) *Elevation.*

Let
$$\left.\begin{aligned} m = 0,\; F_1(r) = 0,\; f_1(r) &= \frac{3V}{2\pi a^3}(a^2-r^2)^{\frac{1}{2}}, \quad a \geq r \\ &= 0 \qquad\qquad\qquad\quad ,\quad a < r \end{aligned}\right\} \tag{47}$$

so that V represents the total volume of fluid initially displaced. Proceeding as in (*i*), we obtain, by (29) and (30), the following expressions for the velocity potential and the surface elevation :

$$\varphi = \frac{3V}{2\pi} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+j} \frac{\Gamma(n+2j+2)}{\Gamma(2n+2)\Gamma(2j+2)(2j+3)} \cdot \frac{g^{n+1} t^{2n+1} a^{2j}}{(z^2+r^2)^{(n+2j+2)/2}} P_{n+2j+1}\left(-\frac{z}{(z^2+r^2)^{\frac{1}{2}}}\right) \tag{48}$$

$$\zeta = \frac{3V}{4\pi^{3/2} r^2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^n \frac{\{\Gamma(n+j+\frac{3}{2})\}^2}{\Gamma(4n+3)\Gamma(j+\frac{5}{2})\Gamma(j+1)} \left(\frac{2gt^2}{r}\right)^{2n+1} \left(\frac{a^2}{r^2}\right)^j, \tag{49A}$$

$$= \frac{V}{\pi^2 r^2} \sum_{n=0}^{\infty} (-1)^n \frac{\{\Gamma(n+\frac{3}{2})\}^2}{\Gamma(4n+3)} \left(\frac{2gt^2}{r}\right)^{2n+1} F\left(n+\tfrac{3}{2},\, n+\tfrac{3}{2},\, \tfrac{5}{2},\, \frac{a^2}{r^2}\right), \qquad r > a: \tag{49B}$$

all of which are true for $r > a$ at $t = 0$.

(γ) *Cylindrical or constant impulse.*

Let
$$\left.\begin{aligned} m = 0, \quad f_1(r) = 0, \quad F_1(r) &= \frac{P}{\pi a^2}, \quad r < a \\ &= 0, \qquad r > a \end{aligned}\right\} \tag{50}$$

so that P represents the total initial impulse on the surface.

$$\therefore \qquad \bar{F}_1(k) = \frac{P}{\pi a k} J_1(ka).$$

And
$$\begin{aligned} [k\bar{F}_1(k)]^{(j)}_{k=0+} &= \frac{(-1)^l P a^{2l}}{\pi . 2^{2l+1}} . \frac{\Gamma(2l+2)}{\Gamma(l+1)\Gamma(l+2)}, \quad j = 2l+1 \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad, \quad j = 2l \end{aligned}$$

where $\qquad l = 0$ or any positive integer.

Equations (29) and (30) now give the velocity potential and the surface elevation corresponding to the disturbance (50):

$$\varphi = \frac{P}{\pi \varrho} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+j} \frac{\Gamma(n+2j+2)}{2^{2j+1}\Gamma(2n+1)\Gamma(j+2)\Gamma(j+1)} \frac{(gt^2)^n a^{2j}}{(z^2+r^2)^{(n+2j+2)/2}} P_{n+2j+1}\left(-\frac{z}{(z^2+r^2)^{\frac{1}{2}}}\right) \tag{51}$$

$$\zeta' = \frac{P}{\pi g \varrho t r^2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} (-1)^{n+j} \frac{\Gamma(n+2j+2)}{2^{2j+1}\Gamma(2n)\Gamma(j+1)\Gamma(j+2)} \left(\frac{gt^2}{r}\right)^n \left(\frac{a^2}{r^2}\right)^j P_{n+2j+1}(0)$$

which are valid only for $r > a$ at $t = 0$. Using (34) and Legendre's duplication formula for Gamma functions, we get

$$\zeta = \frac{P}{\pi^2 g \varrho t r^2} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^n \frac{\{\Gamma(n+j+\frac{3}{2})\}^2}{\Gamma(4n+2)\Gamma(j+1)\Gamma(j+2)} \left(\frac{2gt^2}{r}\right)^{2n+1} \left(\frac{a^2}{r^2}\right)^j \tag{52A}$$

$$= \frac{P}{\pi^2 g \varrho t r^2} \sum_{n=0}^{\infty} (-1)^n \frac{\{\Gamma(n+\frac{3}{2})\}^2}{\Gamma(4n+2)} \left(\frac{2gt^2}{r}\right)^{2n+1} F\left(n+\tfrac{3}{2},\, n+\tfrac{3}{2},\, 2,\, \frac{a^2}{r^2}\right), \qquad r > a. \tag{52B}$$

By using a relation between contiguous hypergeometric functions [Ryshik-Gradstein 1957, 7·237(17)] and the expression for the torus function $D^{m}_{n-\frac{1}{2}}(\cosh \eta)$ in terms of the hypergeometric series, we have, for $r > a$,

$$\zeta = \frac{P}{\pi^{5/2} g \varrho t [ra(r^2-a^2)]^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(4n+2)} \left(\frac{2gt^2}{(r^2-a^2)^{\frac{1}{2}}}\right)^{2n+1} \left[D^{n+1}_{-\frac{1}{2}}\left(\frac{r^2+a^2}{2ar}\right) - \frac{r}{a} D^{n+1}_{\frac{1}{2}}\left(\frac{r^2+a^2}{2ar}\right)\right]. \tag{52C}$$

It may be noted that if the radius a of the circle of initial disturbances tends to zero, all the potential expressions (42), (45), (41) and (51) reduce to the classical solutions (38) or (36) for the velocity potential of the motion due to a concentrated initial impulse or elevation. The same happens with the surface elevation expressions (43*A* & *B*), (46), (49) and (52) which reduce to the expressions (35) or (37) according as the case is one of elevation or of impulse.

ASYMPTOTIC EXPANSION OF THE WAVE INTEGRAL

In this section, it is proposed to obtain a general method of finding the complete asymptotic expansion of the wave integral, in a certain range of large values of r and t, for arbitrary distributions of finitely extended symmetrical initial surface disturbances. Only the first term of this expansion has been obtained before for the general elevation case [Sneddon (1951)] and for the impulse distribution (50) [Hayasi, (1939)] by the application of the method of stationary-phase and for the elevation distribution similar to (50) [Wiedenbauer (1934)] by the saddle-point method. The asymptotic development obtained here will be used to ascertain the solutions of the finitely extended disturbances defined in (41), (44), (47) and its analogous elevation distribution. In this connection, it will be shown that the second term (sometimes the most important term) of the expansion is not wholly contributed by the point of stationary phase nor is it given by the saddle point method, contrary to the conjectures of Hayasi and Wiedenbauer in their particular problems. Lastly, we remark that the process used may be extended to include the cases of asymmetry of the Terazawa type, but this will be reported in future works.

Putting $m = 0$ in (18) and (20A), we have

$$\zeta = (\zeta_1)_{m=0} = \lim_{z\to 0-} \int_0^\infty \Big(-\frac{\sigma}{g\varrho}\bar{F}_1 \sin\sigma t + \bar{f}_1 \cos\sigma t\Big) k J_0(kr) e^{kz}\, dk \tag{53}$$

where

$$\bar{F}_1(k) = \int_0^{a_i} \alpha F_1(\alpha) J_0(k\alpha)\, d\alpha$$

$$\bar{f}_1(k) = \int_0^{a_e} \alpha f_1(\alpha) J_0(k\alpha)\, d\alpha,$$

a_i and a_e being the radii of the circles over which the initial impulse $F_1(r)$ and the initial elevation $f_1(r)$ were present.

(a) Considering first the impulse component ζ_i of ζ in (53), we have

$$-g\varrho\zeta_i = \lim_{z\to 0-} \int_0^{a_i} \sigma F_1(\alpha)\, d\alpha \int_0^\infty k\sigma \sin\sigma t\, e^{kz} J_0(k\alpha) J_0(kr)\, dk \tag{54}$$

on interchanging the orders of α- and k-integrations.

By Neumann's addition theorem for Bessel functions,

$$J_0(kr)J_0(k\alpha) = \frac{1}{\pi}\int_0^\pi J_0(k\lambda)\, d\theta = \frac{1}{\pi^2}\int_0^\pi d\theta \int_0^\pi \cos(k\lambda \sin\varphi)\, d\varphi$$

where $\lambda = (\alpha^2 + r^2 - 2\alpha r\cos\theta)^{\frac{1}{2}}$. (55)

Therefore, by (54),

$$-\pi^2 g\varrho\zeta_i = \lim_{z\to 0-} \int_0^{a_i} \alpha F_1(\alpha)\, d\alpha \int_0^\pi d\theta \int_0^\pi d\varphi \int_0^\infty k\sigma \cos(k\lambda \sin\varphi) \sin\sigma t\, e^{kz}\, dk. \tag{56}$$

Expanding Sin σt in powers of σt, we get

$$\text{the } k\text{-integral} = \sum_{n=0}^{\infty}(-1)^n\frac{g^{n+1}t^{2n+1}}{(2n+1)!}\int_0^{\infty} k^{n+2}e^{kz}\cos(k\lambda\sin\varphi)\,dk,$$

on interchanging the orders of summation and integration.

$$\text{Then, this integral} = \sum_{n=0}^{\infty}(-1)^n\frac{g^{n+1}t^{2n+1}}{(2n+1)!}\cdot\frac{(n+2)!}{(z^2+\lambda^2\sin^2\varphi)^{(n+3)/2}}\cos\left[(n+3)\tan^{-1}\left(\frac{\lambda\sin\varphi}{-z}\right)\right]$$

which, on making $z\to 0-$, becomes

$$= -\frac{1}{t\lambda^2\sin^2\varphi}\sum_{n=0}^{\infty}(-1)^n(2\omega)^{2n+2}\frac{(2n+3)!}{(4n+3)!} \tag{57}$$

where
$$\omega = \frac{gt^2}{2\lambda\sin\varphi}. \tag{58}$$

Introducing the functions $L(\omega)$ and $M(\omega)$ by the series

$$\left.\begin{aligned} L(\omega) &= \sum_{n=0}^{\infty}(-1)^n\frac{\omega^{2n+1}2^{2n}(2n)!}{(4n+1)!} \\ M(\omega) &= \sum_{n=0}^{\infty}(-1)^n\frac{\omega^{2n+2}2^{2n+1}(2n+1)!}{(4n+3)!} \end{aligned}\right\} \tag{59}$$

so that [Ryshik-Gradstein, 1957, 6·233(2,3)]

$$L(\omega) = \frac{\omega^{\frac{1}{2}}}{2}\left[\cos\frac{\omega}{2}\int_0^{\infty}\cos\frac{u}{2}\,\frac{du}{u^{\frac{1}{2}}} + \sin\frac{\omega}{2}\int_0^{\infty}\sin\frac{u}{2}\,\frac{du}{u^{\frac{1}{2}}}\right]$$

$$= (\pi\omega)^{\frac{1}{2}}\left[\cos\frac{\omega}{2}\,C\left(\frac{\omega}{2}\right) + \sin\frac{\omega}{2}\,S\left(\frac{\omega}{2}\right)\right] \tag{60}$$

and

$$M(\omega) = \frac{\omega^{\frac{1}{2}}}{2}\left[\sin\frac{\omega}{2}\int_0^{\infty}\cos\frac{u}{2}\,\frac{du}{u^{\frac{1}{2}}} - \cos\frac{\omega}{2}\int_0^{\infty}\sin\frac{u}{2}\,\frac{du}{u^{\frac{1}{2}}}\right]$$

$$= (\pi\omega)^{\frac{1}{2}}\left[\sin\frac{\omega}{2}\,C\left(\frac{\omega}{2}\right) - \cos\frac{\omega}{2}\,S\left(\frac{\omega}{2}\right)\right] \tag{61}$$

where $C(x)$ and $S(x)$ denote Fresnel's integrals defined by

$$[C(x),\, S(x)] = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_0^{x}\frac{[\cos t, \sin t]}{t^{\frac{1}{2}}}\,dt,$$

we find the series on the *r.h.s.* of (57) $= \frac{1}{2}[3M(\omega) + 6\omega L(\omega) + \omega^2\{1 - M(\omega)\}]$

From (56) now,

$$\pi^2 g\varrho\zeta_t = \frac{1}{2t}\int_0^{a_1}\alpha F_1(\alpha)\,d\alpha\int_0^{\pi}\frac{d\theta}{\lambda^2}\int_0^{\pi}\frac{d\varphi}{\sin^2\varphi}\,[3M(\omega) + 6\omega L(\omega) + \omega^2\{1 - M(\omega)\}] \tag{62}$$

In the approximations that follow, the range of large values of r and t is defined by the conditions:

$$\frac{r}{a_i} \gg 1, \ \frac{gt^2}{4r} \gg 1, \ \frac{r}{a_i} \gg \frac{a_i g t^2}{4r^2}. \tag{63}$$

Then $\omega \gg 1$. Using the asymptotic expansions of Fresnel's integrals [Erdélyi, 1953, 9.10(8,9)], we find those of the functions L and M to be

$$\left.\begin{aligned} L(\omega) &= \frac{(\pi\omega)^{\frac{1}{2}}}{2}\left(\sin\frac{\omega}{2}+\cos\frac{\omega}{2}\right)-\left\{\frac{1}{\omega}-\frac{1.3.5}{\omega^3}+\dots\right\} \\ M(\omega) &= \frac{(\pi\omega)^{\frac{1}{2}}}{2}\left(\sin\frac{\omega}{2}-\cos\frac{\omega}{2}\right)+\left\{1-\frac{1.3}{\omega^2}+\frac{1.3.5.7}{\omega^4}-\dots\right\} \end{aligned}\right\} \tag{64}$$

Inserting these in (62), we next propose to retain, in ζ_i, up to the terms of the second order of smallness as indicated by (63). The first term within the bracket in (62) can then be neglected, since it contributes a term $O(r^{-3})$ to ζ_i (vide 1st term of (74)).

Putting
$$\mu = \frac{gt^2}{4\lambda}, \tag{65}$$

we have then

$$\begin{aligned} \text{the } \varphi\text{-integral of (62)} &\approx 3\pi^{\frac{1}{2}}(2\mu)^{3/2}\int_0^{\pi}\frac{1}{\sin^{7/2}\varphi}\left\{\sin\left(\frac{\mu}{\sin\varphi}\right)+\cos\left(\frac{\mu}{\sin\varphi}\right)\right\}d\varphi \\ &\quad+\frac{\pi^{\frac{1}{2}}}{2}(2\mu)^{5/2}\int_0^{\pi}\frac{d\varphi}{\sin^{9/2}\varphi}\left\{\cos\left(\frac{\mu}{\sin\varphi}\right)-\sin\left(\frac{\mu}{\sin\varphi}\right)\right\} \\ &= 6\pi^{\frac{1}{2}}(2\mu)^{3/2}\int_0^{\pi/2}\frac{d\varphi}{\sin^{7/2}\varphi}\left\{\sin\left(\frac{\mu}{\sin\varphi}\right)+\cos\left(\frac{\mu}{\sin\varphi}\right)\right\} \\ &\quad+\pi^{\frac{1}{2}}(2\mu)^{5/2}\int_0^{\pi/2}\frac{d\varphi}{\sin^{9/2}\varphi}\left\{\cos\left(\frac{\mu}{\sin\varphi}\right)-\sin\left(\frac{\mu}{\sin\varphi}\right)\right\} \\ &= 6\pi^{\frac{1}{2}}(2\mu)^{3/2}\int_0^{\infty}\cosh^{5/2}t\{\sin(\mu\cosh t)+\cos(\mu\cosh t)\}dt \\ &\quad+\pi^{\frac{1}{2}}(2\mu)^{5/2}\int_0^{\infty}\cosh^{7/2}t\{\cos(\mu\cosh t)-\sin(\mu\cosh t)\}dt \\ &= 3\pi\mu^2\int_0^{\infty}[3\cosh t+\cosh 3t].[J_{\frac{1}{2}}(\mu\cosh t)+J_{-\frac{1}{2}}(\mu\cosh t)]dt \\ &\quad+\pi\mu^3\int_0^{\infty}[\tfrac{3}{2}+2\cosh 2t+\tfrac{1}{2}\cosh 4t][J_{-\frac{1}{2}}(\mu\cosh t)-J_{\frac{1}{2}}(\mu\cosh t)]dt. \end{aligned} \tag{66}$$

Using the result [Erdélyi, (1953), 7·14·2(65)]

$$\int_0^\infty J_{\alpha+\beta}(2z\cosh t)\cosh(\alpha-\beta)t\,dt = -\frac{\pi}{4}\left[J_\alpha(z)Y_\beta(z)+J_\beta(z)Y_\alpha(z)\right],\ z>0, \tag{67}$$

we have

$$\begin{aligned}\text{the } \varphi\text{-integral of (62)} \approx & -\tfrac{3}{4}\pi^2\mu^3\big[3\{J_{\frac34}Y_{-\frac14}+J_{-\frac14}Y_{\frac34}+J_{\frac14}Y_{-\frac34}+J_{-\frac34}Y_{\frac14}\}\\ &\qquad +\{J_{\frac74}Y_{-\frac54}+J_{-\frac54}Y_{\frac74}+J_{\frac54}Y_{-\frac74}+J_{-\frac74}Y_{\frac54}\}\big]\\ & -(\pi^2/4)\mu^3\big[3\{J_{-\frac14}Y_{-\frac14}-J_{\frac14}Y_{\frac14}\}+2\{J_{\frac34}Y_{-\frac54}+J_{-\frac54}Y_{\frac34}-J_{\frac54}Y_{-\frac34}-J_{-\frac34}Y_{\frac54}\}\\ &\qquad +\tfrac12\{J_{\frac74}Y_{-\frac94}+J_{-\frac94}Y_{\frac74}-J_{\frac94}Y_{-\frac74}-J_{-\frac74}Y_{\frac94}\}\big],\end{aligned} \tag{68}$$

$\mu/2$ being the argument of all the Bessel's and Neumann's functions written above. Using the asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$ for large values of z [Erdélyi, (1953), 7·13·1(3,4)] and retaining only those terms in the asymptotic development of the *r.h.s.* of (68) which contribute up to the second order terms in ζ_i, we find

$$\text{the } \varphi\text{-integral of (62)} \approx 12.2^{\frac14}\pi\,\mu\cos\mu - \frac{\pi}{2^{\frac14}}\,\mu^2\left(8\sin\mu+13\frac{\cos\mu}{\mu}\right).$$

Consequently,

$$\text{the } \theta\text{-integral of (62)} \approx \frac{\pi g t^2}{2^{3/2}}\int_0^\pi \frac{d\theta}{\lambda^3}\left\{-\frac{gt^2}{\lambda}\sin\frac{gt^2}{4\lambda}+\frac{11}{2}\cos\frac{gt^2}{4\lambda}\right\} \tag{69}$$

Expanding λ^{-1}, λ^{-3}, λ^{-4} in powers of r^{-1} and retaining only those terms in the asymptotic expansion of (69) which contribute up to the second order terms in ζ_i, we get

$$\begin{aligned}\text{the } \theta\text{-integral of (62)} \approx & \frac{\pi g t^2}{2^{3/2}}\Big[-\frac{gt^2}{r^4}\int_0^\pi\Big(1+\frac{4\alpha}{r}\cos\theta\Big)\Big\{\sin\Big[\frac{gt^2}{4r}\Big(1+\frac{\alpha}{r}\cos\theta\Big)\Big]+\frac{\alpha^2 g t^2}{16r^3}(1+3\cos2\theta)\times\\ &\cos\Big[\frac{gt^2}{4r}\Big(1+\frac{\alpha}{r}\cos\theta\Big)\Big]\Big\}d\theta+\frac{11}{2r^3}\int_0^\pi\cos\Big\{\frac{gt^2}{4r}\Big(1+\frac{\alpha}{r}\cos\theta\Big)\Big\}d\theta\Big]\\ = & \frac{\pi^2 g t^2}{2^{3/2}}\Big[-\frac{gt^2}{r^4}\Big\{J_0\Big(\frac{\alpha g t^2}{4r^2}\Big)\sin\frac{gt^2}{4r}+\Big(\frac{4\alpha}{r}J_1\Big(\frac{\alpha g t^2}{4r^2}\Big)+\frac{\alpha^2 g t^2}{16r^4}\Big[J_0\Big(\frac{\alpha g t^2}{4r^2}\Big)-3J_2\Big(\frac{\alpha g t^2}{4r^2}\Big)\Big]\Big)\cos\frac{gt^2}{4r}\Big\}\\ &+\frac{11}{2r^3}J_0\Big(\frac{\alpha g t^2}{4r^2}\Big)\cos\frac{gt^2}{4r}\Big]\end{aligned}$$

The relation (62) therefore becomes

$$\begin{aligned}\zeta_i \approx -\frac{t}{2^{5/2}gr^3}\int_0^{\alpha_i}\alpha F_1(\alpha)d\alpha\Big[&\frac{gt^2}{r}J_0\Big(\frac{\alpha g t^2}{4r^2}\Big)\sin\frac{gt^2}{4r}+\Big\{\frac{4\alpha g t^2}{r^2}J_1\Big(\frac{\alpha g t^2}{4r^2}\Big)\\ &+\frac{\alpha^2 g^2 t^4}{16r^4}\Big[J_0\Big(\frac{\alpha g t^2}{4r^2}\Big)-3J_2\Big(\frac{\alpha g t^2}{4r^2}\Big)\Big]-\frac{11}{2}J_0\Big(\frac{\alpha g t^2}{4r^2}\Big)\Big\}\cos\frac{gt^2}{4r}\Big]\end{aligned} \tag{70}$$

Expressing the different integrals in terms of $\overline{F}_1\left(\frac{gt^2}{4r^2}\right)$ and its derivates $\overline{F}_1'\left(\frac{gt^2}{4r^2}\right)$ and $F_1''\left(\frac{gt^3}{4r^2}\right)$ by the help of the recurrence relations between Bessel's functions and their derivates, we get finally,

$$\zeta_i = -\frac{gt^3}{2^{5/2}\varrho r^4}\overline{F}_1\left(\frac{gt^2}{4r^2}\right)\sin\frac{gt^2}{4r} + \frac{t}{2^{7/2}\varrho r^3}\left[11\overline{F}_1\left(\frac{gt^2}{4r^2}\right) + 7\frac{gt^2}{r^2}\overline{F}_1'\left(\frac{gt^2}{4r^2}\right)\right.$$

$$\left. + \frac{g^2t^4}{2r^4}\overline{F}_1''\left(\frac{gt^3}{4r^2}\right)\right]\cos\frac{gt^2}{4r} + O(r^{-3}) \qquad (71)$$

under the restrictions (63).

(*b*) Proceeding with the elevation component ζ_e of ζ in (53) in the same way, we have, corresponding to (56)

$$\pi^2\zeta_e = \lim_{z\to 0-}\int_0^{a_0}\alpha f_1(\alpha)d\alpha\int_0^{\pi}d\theta\int_0^{\pi}d\varphi\int_0^{\infty}k\cos\sigma t e^{kz}\cos(k\lambda\sin\varphi)dk. \qquad (72)$$

And

$$\lim_{z\to 0-}(\text{the } k\text{-integral of } (72)) = \frac{1}{\lambda^2\sin^2\varphi}\sum_{n=0}^{\infty}(-1)^{n+1}(2\omega)^{2n}\frac{(2n+1)!}{(4n)!}$$

$$= \frac{1}{2\lambda^2\sin^2\varphi}[3M(\omega) + \omega L(\omega) - 2].$$

Corresponding to (62), we have now

$$\pi^2\zeta_e = \tfrac{1}{2}\int_0^{a_0}\alpha f_1(\alpha)d\alpha\int_0^{\pi}\frac{d\theta}{\lambda^2}\int_0^{\pi}\frac{d\varphi}{\sin^2\varphi}[3M(\omega) + \omega L(\omega) - 2]. \qquad (73)$$

Retaining, in ζ_e, only up to the terms of the second order of smallness in the scale of (63), we get, in the same way as before,

$$\text{the } \varphi\text{-integral of } (73) = -\frac{3\pi^2\mu}{8}[2(J_{\frac{1}{4}}Y_{\frac{1}{4}} - J_{-\frac{1}{4}}Y_{-\frac{1}{4}}) + J_{\frac{5}{4}}Y_{-\frac{3}{4}} + J_{-\frac{3}{4}}Y_{\frac{5}{4}} - J_{\frac{3}{4}}Y_{-\frac{5}{4}} - J_{-\frac{5}{4}}Y_{\frac{3}{4}}]$$

$$-\frac{\pi^2\mu^2}{8}[3\{J_{\frac{3}{4}}Y_{-\frac{1}{4}} + J_{-\frac{1}{4}}Y_{\frac{3}{4}} + J_{\frac{1}{4}}Y_{-\frac{3}{4}} + J_{-\frac{3}{4}}Y_{\frac{1}{4}}\}$$

$$+\{J_{\frac{7}{4}}Y_{-\frac{5}{4}} + J_{-\frac{5}{4}}Y_{\frac{7}{4}} + J_{\frac{5}{4}}Y_{-\frac{7}{4}} + J_{-\frac{7}{4}}Y_{\frac{5}{4}}\}] \qquad (74)$$

$\mu/2$ being the argument of all the Bessel's and Neumann's functions written above. Using the asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$ for large values of z, we have, to the above degree of approximation, the φ-integral of (73) $\approx 3.2^{\frac{1}{2}}\pi \sin\mu + 2.2^{\frac{1}{2}}\pi\mu\cos\mu - \frac{9\pi}{2\,2^{\frac{1}{2}}}\sin\mu$

Consequently,

the θ-integral of (73) $\approx \frac{\pi}{2^{\frac{1}{2}}}\int_0^{\pi}\frac{d\theta}{\lambda^2}\left(\frac{gt^2}{\lambda}\cos\mu + \frac{3}{2}\sin\mu\right)$

$$\approx \frac{\pi g t^2}{2^{\frac{1}{2}}r^3}\int_0^{\pi} d\theta\left(1+\frac{3\alpha}{r}\cos\theta\right)\left[\cos\left\{\frac{gt^2}{4r}\left(1+\frac{\alpha}{r}\cos\theta\right)\right\} - \frac{\alpha^2 g t^2}{16r^3}(1+3\cos 2\theta)\sin\right)\frac{gt^2}{4r}\times$$

$$\left(1+\frac{\alpha}{r}\cos\theta\right)\Big\}\Big] + \frac{3\pi}{2.2^{\frac{1}{2}}r^2}\int_0^{\pi}\sin\left\{\frac{gt^2}{4r}\left(1+\frac{\alpha}{r}\cos\theta\right)\right)d\theta$$

$$= \frac{\pi^2 g t^2}{2^{\frac{1}{2}}r^3}\left[J_0\left(\frac{\alpha g t^2}{4r^2}\right)\cos\frac{gt^2}{4r} + \left\{-\frac{3\alpha}{r}J_1\left(\frac{\alpha g t^2}{4r^2}\right) - \frac{\alpha^2 g t^2}{16r^3}\left[J_0\left(\frac{\alpha g t}{4r^2}\right) - 3J_2\left(\frac{\alpha g t^2}{4r^2}\right)\right]\right\}\sin\frac{gt^2}{4r}\right]$$

$$+\frac{3\pi^2}{2.2^{\frac{1}{2}}r^2}J_0\left(\frac{\alpha g t^2}{4r^2}\right)\sin\frac{gt^2}{4r}\,.$$

∴ from (73), we have

$$\zeta_e \approx \frac{gt^2}{2^{3/2}r^3}\int_0^{a_e}\alpha f_1(\alpha)d\alpha\left[J_0\left(\frac{\alpha g t^2}{4r^2}\right)\cos\frac{gt^2}{4r} - \left\{\frac{3\alpha}{r}J_1\left(\frac{\alpha g t^2}{4r^2}\right) - \frac{3r}{2gt^2}J_0\left(\frac{\alpha g t^2}{4r^2}\right)\right.\right.$$

$$\left.\left.+\frac{\alpha^2 g t^2}{16r^3}\left[J_0\left(\frac{\alpha g t^2}{4r^2}\right) - 3J_2\left(\frac{\alpha g t^2}{4r^2}\right)\right]\right\}\sin\frac{gt^2}{4r}\right]$$

and, as before, finally

$$\zeta_e = \frac{gt^2}{2^{3/2}r^3}\left[\bar f_1\left(\frac{gt^2}{4r^3}\right)\cos\frac{gt^2}{4r} + \left\{\frac{3r}{2gt^2}\bar f_1\left(\frac{gt^2}{4r^2}\right) + \frac{5}{2r}\bar f_1'\left(\frac{gt^2}{4r^2}\right) + \frac{gt^2}{4r^3}\bar f_1''\left(\frac{gt^2}{4r^2}\right)\right\}\sin\frac{gt^2}{4r}\right]$$

$$+O(r^{-3}) \tag{75}$$

under the restrictions (63) where a_e now replaces a_i.

The resulting surface elevation is now, by (53),

$$\zeta = \zeta_i + \zeta_e. \tag{76}$$

The first term of (75) without the third restriction of (63) agrees with that obtained by Sneddon (1951) [32·6(89)] for a general initial surface elevation without initial impulse on the assumption of Stokes-Kelvin method of stationary phase. It can be shown that the first term of (71) without the third restriction of (63) is also the same as what may be obtained by applying the stationary phase hypothesis to the exact integral for ζ_i.

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In the following, $t' = \frac{(agt^2)^{\frac{1}{2}}}{2r}$ and $r' = \frac{r}{a}$. (77)

(*a*) *Paraboloidal Impulse* (41). In this case,

$$\bar{F}_1(k) = \frac{4P}{\pi a^2} \cdot \frac{J_2(ka)}{k^2} .$$

The equation (71), therefore, gives

$$\zeta = -\frac{2^{5/2}P}{\pi\varrho g^{\frac{1}{2}}a^{5/2}t'r'}J_2(t'^2)\sin(r't'^2) + \frac{2^{\frac{1}{2}}Pt'}{\pi\varrho g^{\frac{1}{2}}a^{5/2}r'^2}\left[\frac{11}{2}\frac{J_2(t'^2}{t'^4} - 3J_2(t'^2) + J_4(t'^2)\right]\cos(r\,t'^2) - O(r'^{-3}) \tag{78}$$

under the restrictions (63):

$$r' \gg 1,\ r't'^2 \gg 1,\ r' \gg t'^2. \tag{79}$$

(*b*) *Ellipsoidal Elevation* (47). Here

$$f_1(k) = \frac{3V}{2\pi a^3}\cdot\frac{\sin ka - ak\cos ka}{k^3} .$$

From eqn. (75), therefore,

$$\zeta = \frac{3V}{2^{\frac{1}{2}}\pi a^2 t'^2 r'}\left(\frac{\sin t'^2}{t'^2} - \cos t'^2\right)\cos(r't'^2)$$

$$+ \frac{3.2^{\frac{1}{2}}V}{\pi a^2 r'^2}\left[\left(\frac{39}{16}\cdot\frac{1}{t'^4} - \frac{5}{4}\right)\frac{\sin t'^2}{t'^2} - \left(\frac{39}{16}\cdot\frac{1}{t'^4} - \frac{1}{2}\right)\cos t'^2\right]\sin(r't'^2) + O(r'^{-3}) \tag{80}$$

under the restrictions (79).

(*c*) *Ellipsoidal Impulse* (44). We have

$$\bar{F}_1(k) = \frac{3P}{2\pi a^3}\cdot\frac{\sin ka - ka\cos ka}{k^3} .$$

The equation (71) gives

$$\zeta = -\frac{3P}{2^{\frac{1}{2}}\pi\varrho g^{\frac{1}{2}}a^{5/2}t'r'}\left(\frac{\sin t'^2}{t'^2}. - \cos t'^2\right)\sin(r't'^2)$$

$$+ \frac{3.2^{\frac{1}{2}}Pt'}{\pi\varrho g^{\frac{1}{2}}a^{5/2}r'^2}\left[\left(\frac{23}{16}\,\frac{1}{t'^4} - \frac{3}{4}\right)\frac{\sin t'^2}{t'^2} - \left(\frac{23}{16}\,\frac{1}{t'^4} - 8\right)\cos t'^2\right]\cos(r't'^2) + O(r'^{-3}) \tag{81}$$

under the restrictions (79).

(*d*) *Cylindrical Impulse* (50). Here

$$F_1(k) = \frac{P}{\pi ak}J_1(ka).$$

Therefore eqn. (71) gives

$$\zeta = -\frac{2^{\frac{1}{2}}Pt'}{\pi\varrho g^{\frac{1}{2}}a^{5/2}r'}J_1(t'^2)\sin(r't'^2)$$

$$+ \frac{Pt'}{2^{\frac{1}{2}}\pi\varrho g^{\frac{1}{2}}a^{5/2}r'^2}\left[J_0(t'^2) + \left(\frac{3}{4}\cdot\frac{1}{t'^2} - 2t'^2\right)J_1(t'^2)\right]\cos(r't'^2) + O(r'^{-3}) \tag{82}$$

under the restrictions (79).

(*e*) *Cylindrical Elevation.* Let

$$\left.\begin{aligned} m = 0,\ F_1(r) = 0,\ f_1(r) &= \frac{V}{\pi a^2},\ r < a \\ &= 0, \quad r > a \end{aligned}\right\} \tag{83}$$

so that V represents the total volume of fluid initially displaced. Hence

$$\overline{f_1}(k) = \frac{V}{\pi a k} J_1(ka).$$

$\therefore$ eqn. (75) gives

$$\zeta = \frac{2^{\frac{1}{2}} V}{\pi a^2 r'} J_1(t'^2) \cos(r't'^2) + \frac{V}{2^{\frac{1}{2}} \pi a^2 r'^2} \left[\frac{11}{4} \frac{J_1(t'^2)}{t'^2} - 2t'^2 J_1(t'^2) - J_0(t'^2) \right] \sin(r't'^2) + O(r'^{-3}) \tag{84}$$

under the restrictions (79).

The first term on the r.h.s. of (82) agrees with that obtained by Hayasi (1939) on the application of the method of stationary phase to the impulse problem (*d*). Also, the first term of (84) agrees with that found by Wiedenbauer (1934) by the saddle point method for the elevation problem (*e*). The second terms of (82) and (84), however, differ from their expressions perhaps because the second and higher order terms do not wholly arise from the neighbourhood of the point of stationary phase and are probably not completely given by the saddle point method as supposed by Hayasi and Wiedenbauer in their respective problems.

It may be noted that if $a \to 0$, the first terms of the impulse expressions (78), (81) and (82) reduce to

$$\zeta = -\frac{Pgt^3}{2^{7/2} \pi \varrho r} \sin \frac{gt^2}{4r}$$

and those of the elevation expressions (80) and (84) reduce to

$$\zeta = \frac{Vgt^2}{2^{5/2} \pi r^3} \cos \frac{gt^2}{4r}.$$

which are the well-known Cauchy-Poisson solutions for concentrated disturbances.

The various consequences of the results obtained here will be discussed in subsequent parts of this work.

In conclusion, I wish to express my thanks to Prof. B. B. Sen for his kind guidance and helpful suggestions in course of this work.

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UNSTEADY FLOW OF A VISCOUS HOMOGENEOUS INCOMPRESSIBLE FLUID IN A CIRCULAR PIPE OF UNIFORM CROSS-SECTION

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1. **Introduction.** The motion of a viscous homogeneous incompressible fluid in a circular pipe of uniform cross-section has been studied by many workers in the field of hydrodynamics. In the case of steady flow of a viscous homogeneous incompressible fluid, the pressure gradient along the axis of the pipe comes out to be a constant. In such a case we get the famous Poise uille flow for which the velocity of the liquid along the axis of the pipe has parabolic distribution with respect to the ordinate perpendicular to the axis of the pipe·

But in the case when the flow is unsteady the pressure gradient along the axis of the pipe is not necessarily a constant. In general, it is a function of time alone.

In the present paper we have discussed the unsteady flow of a viscous homogeneous incompressible fluid, by assuming the pressure gradint to be a suitable function of time. In particular, if the pressure gradient is constant, the velocity distribution becomes parabolic after infinite time. On the other hand if the pressure gradient is not a constant but vanishes as $t \to \infty$, the fluid comes to rest at $t = \infty$. It has been remarked that the same solution is valid for non-Newtonian fluids also when the coefficients of shear and cross viscosity are constants.

2. **Formulation of the Problem.** Let a viscous homogeneous incompressible fluid be in motion in a circular pipe of radius a. Initially the liquid is assumed to be at rest. Let the axis of pipe be the x-axis and the coordinate perpendicular to it measured along a meridian plane passing through the x-axis, be y. The motion starts due to the pressure gradient, say $\varrho f(t)$, along the axis of the pipe. Let u and v be the velocity components along the axes of x and y, respectively. From the symmetry of the problem u is independent of x and v is zero. The equations of motion are

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right) - f(t) \tag{2.1}$$

$$0 = \frac{\partial p}{\partial y}, \tag{2.2}$$

where the body forces are assumed to be absent and $\frac{1}{\varrho}\frac{\partial p}{\partial x} = f(t)$. The other symbols used above have their usual meanings.

The boundary and initial conditions are

$$\left.\begin{array}{lll} (i) & u(a,\,t) = 0 \quad \text{for} \quad t \geq 0 \\ (ii) & u(y,\,0) = 0 \quad \text{at} \quad t = 0 \;\; 0 \leq y \leq a \end{array}\right\} \tag{2.3}$$

3. **Solution of the Problem.** Applying Laplace transform and using (2.3), equation (2.1) becomes

$$s\bar{u}(y,\,s) = \nu\left(\frac{d^2}{dy^2}\,\bar{u}(y,\,s) + \frac{1}{y}\frac{d}{dy}\bar{u}(y,\,s)\right) - \bar{f}(s) \tag{3.1}$$

where
$$\bar{u}(y,\,s) = \int_0^\infty e^{-st}u(y,\,t)dt$$

and
$$\bar{f}(s) = \int_0^\infty e^{-st}f(t)dt$$

We assume the function $\bar{f}(s)$ to be of the type $P(s)/F(s)$ where both $P(s)$ and $F(s)$ are single-valued.

$P(s)$ is regular in the s-plane and therefore the poles of $\bar{f}(s)$ are the zeros of $F(s)$. The degree of $P(s)$ is less than that of $F(s)$. Further-more for the simplicity of calculation we assume that the zeros of $F(s)$ are all simple, isolated, finite and lie on the negative real axis.

Now the boundary and initial conditions are

$$\left.\begin{array}{lll} (i) & \bar{u}(a,\,s) = 0 \quad \text{for} \quad t \geq 0 \\ (ii) & \bar{u}(y,\,0) = 0 \quad \text{at} \quad t = 0,\;\; 0 \leq y \leq a \end{array}\right\} \tag{3.2}$$

Solving equation (3.1), we get

$$\bar{u}(y,\,s) = AJ_0(iy\sqrt{s/\nu}) + BY_0(iy\sqrt{s/\nu}) - \frac{\bar{f}(s)}{s}$$

The velocity of the liquid along the axis $y = 0$ is finite and so also is its Laplace transform. Therefore B must be zero. At the boundary, $y = a$, the liquid is at rest. Thus $\bar{u}(a,\,s) = 0$, and we get

$$A = \frac{\bar{f}(s)}{sJ_0(ia\sqrt{s/\nu})} = \frac{P(s)}{sF(s)J_0(ia\sqrt{s/\nu})}$$

Hence
$$\bar{u}(y,\,s) = \frac{P(s)}{sF(s)}\left\{\frac{J_0(iy\sqrt{s/\nu})}{J_0(ia\sqrt{s/\nu})} - 1\right\} \tag{3.3}$$

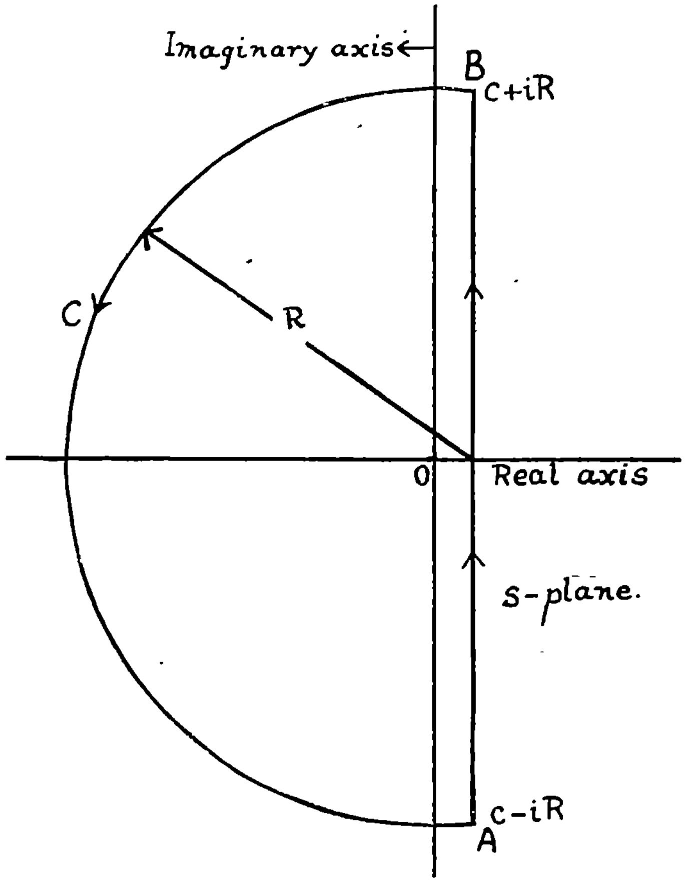
From the inversion formula for Laplace transform, we have

$$u(y, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{P(s)e^{st}}{sF(s)} \frac{J_0(iy\sqrt{s/\nu})-J_0(ia\sqrt{s/\nu})}{J_0(ia\sqrt{s/\nu})}ds \tag{3.4}$$

The integration is to be performed along a straight line $x = c$ from $-i\infty$ to $+i\infty$. All the poles of the integrand in (3.4) are the zeros of $J_0(ia\sqrt{s/\nu})$, say $\lambda_1, \lambda_2, \ldots\lambda_n, \ldots$, and the zeros of $F(s)$ say $-b_1, -b_2 \ldots, -b_n, \ldots$, where all b's are real and positive. Thus we write

$$s_n = -\frac{\nu\lambda_n^2}{a^2}, \text{ the } n^{\text{th}} \text{ zero of } J_0(ia\sqrt{s/\nu})$$

and $s_q = -b_q$, the q^{th} zero of $F(s)$.*



All the poles of the integrand in this case lie to the left of the line $x = c$. Now consider the contour as shown in Fig. 1. The radius R of the semi-circle BCA tends to infinity so that all the poles of the given integrand lie inside this contour.

---

*It is assumed that $P(s)$ does not have a zero in common with $J_0(ia\sqrt{s/\nu})$ and $F(s)$ does not haev a zero in common with $(J_0(iy\sqrt{s/\nu})-J_0(ia\sqrt{s/\nu}))$

The integral along the closed path is then represented as the sum along the two segments

$$\int\limits_{\substack{ABCA\\(\text{circuit})}} = \int\limits_{\substack{AB\\(\text{chord})}} + \int\limits_{\substack{BCA\\(\text{arc})}}$$

which can be written as

$$\int\limits_{\substack{AB\\(\text{chord})}} = \int\limits_{\substack{ABCA\\(\text{circuit})}} - \int\limits_{\substack{BCA\\(\text{arc})}}$$

If we let $R \to \infty$, we get $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}$

$$= \lim_{R\to\infty} \frac{1}{2\pi i} \int\limits_{\substack{ABCA\\(\text{circuit})}} - \frac{1}{2\pi i} \lim_{R\to\infty} \int\limits_{\substack{BCA\\(\text{arc})}} \tag{3.5}$$

$$\text{Now } \lim_{s\to\infty} \frac{P(s)}{sF(s)} \frac{J_0(iy\sqrt{s/\nu}) - J_0(ia\sqrt{s/\nu})}{J_0(ia\sqrt{s/\nu})} = \lim_{s\to\infty} \frac{P(s)}{sF(s)} \frac{I_0(y\sqrt{s/\nu}) - I_0(a\sqrt{s/\nu})}{I_0(a\sqrt{s/\nu})}$$

$$\text{or} \qquad \lim_{s\to\infty} \frac{P(s)}{sF(s)} \frac{I_0(y\sqrt{s/\nu}) - I_0(a\sqrt{s/\nu})}{I_0(a\sqrt{s/\nu})} \sim \lim_{s\to\infty} \frac{P(s)}{sF(s)} \left\{ \left(\frac{a}{y}\right)^{\frac{1}{2}} e^{-(a-y)(\sqrt{s/\nu})} - 1 \right\} \to 0,$$

since $a \geq y$ and the degree of $F(s)$ is greater then that of $P(s)$

Therefore from Jordan's Lemma (Thomson, 1950) the second integral on the right hand side of (3.5) is zero and the integral inside the closed contour is equal to the sum of the residues at the poles of the integrand. Thus

$$u(y, t) = \frac{1}{2\pi i} [2\pi i \text{ (sum of the residues at the zeros of } J_0(ia\sqrt{s/\nu}))$$

$$+ 2\pi i \text{ (sum of the residues at the zeros of } F(s))]$$

$$= -2\sum_{n=1}^{\infty} \frac{e^{-\nu\lambda^2 t/a^2}}{\lambda_n J_1(\lambda_n)} \cdot \frac{J_0\left(\frac{y}{a}\lambda_n\right) P(-\nu\lambda_n^2/a^2)}{F(-\nu\lambda_n^2/a^2)} + \sum_{q=1}^{m} \frac{P(-b_q)e^{-b_q t}}{b_q F'(-b_q)} \left\{ 1 - \frac{J_0\left(\frac{y}{ak_q}\right)}{J_0(k_q)} \right\} \tag{3.6}$$

where $k_q = (a^2 b_q/\nu)^{\frac{1}{2}}$ and $-\nu\lambda_n^2/a^2$ is not a zero of $F(s)$ for any λ_n.

If we put $\xi = y/a$, in (3.6) the expression for $u(y, t)$ becomes

$$u(\xi, t) = \sum_{q=1}^{m} \frac{P(-b_q)e^{-b_q t}}{b_g F'(-b_q)} \left\{ 1 - \frac{J_0(\xi k_q)}{J_0(k_q)} \right\} - 2\sum_{n=1}^{\infty} \frac{P(-\nu\lambda_n^2/a^2)}{F(-\nu\lambda_n^2/a^2)} \frac{J_0(\xi\lambda_n)}{\lambda_n J_1(\lambda\ )} e^{-\nu\lambda_n^2 t/a^2} \tag{3.7}$$

$$0 \leq \xi \leq 1.$$

At $t = 0$, $u(\xi, 0) = 0$, therefore we must have

$$\sum_{q=1}^{m} \frac{P(-b_q)}{b_q F'(-b_q)} \left\{1 - \frac{J_0(\xi k_q)}{J_0(k_q)}\right\} = 2\sum_{n=1}^{\infty} \frac{P(-\nu\lambda_n^2/a^2)}{F(-\nu\lambda_n^2/a^2)} \frac{J_0(\lambda_n \xi)}{\lambda_n J_1(\lambda_n)} \tag{3.8}$$

The equality (3.8) can be established in the following manner. Let us expand the left hand side of (3.8) in a Fourier-Bessel series (Watson, 1941), *i.e.*, let

$$\sum_{q=1}^{m} \frac{P(-b_q)}{b_q F'(-b_q)} \left\{1 - \frac{J_0(\xi k_q)}{J_0(k_q)}\right\} = \sum_{n=1}^{\infty} B_n J_0(\xi \lambda_n) \tag{3.9}$$

where
$$B_n = \frac{2}{J_1(\lambda_n)} \int_0^1 \xi \left\{\sum_{q=1}^{m} \frac{P(-b_q)}{b_q F'(-b_q)} \left(1 - \frac{J_0(\xi k_q)}{J_0(k_q)}\right)\right\} \times J_0(\xi\lambda_n) d\xi$$

and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \ldots$

are the zeros of $J_0(z)$. To evaluate B_n we proceed as follows:

$$\begin{aligned}
B_n &= \frac{2}{J_1^2(\lambda_n)} \sum_{q=1}^{m} \frac{P(-b_q)}{b_q F'(-b_q)} \left\{\int_0^1 \xi J_0(\lambda_n \xi) d\xi - \int_0^1 \frac{J_0(\xi\lambda_n) J_0(\xi k_q)}{J_0 k_q} d\xi\right\} \\
&= \frac{2}{J_1^2(\lambda_n)} \sum_{q=1}^{m} \frac{P(-b_q)}{b_q F'(-b_q)} \left\{\frac{J_1(\lambda_n)}{\lambda_n} - \frac{(\lambda_n J_1(\lambda_n)}{(\lambda_n^2 - k_q^2)}\right\} \\
&= \frac{2}{\lambda_n J_1(\lambda_n)} \sum_{q=1}^{m} \frac{P(-b_q)}{\nu F'(-b_q)} \frac{a^2}{(k_q^2 - \lambda_n^2)} \\
&= \frac{2}{\lambda_n J_1(\lambda_n)} \sum_{q=1}^{m} \frac{P(-b_q)}{F'(-b_q)\{b_q - \nu\lambda_n^2/a^2\}} \\
&= \frac{2P(-\nu\lambda_n^2/a^2)}{\lambda_n J_1(\lambda_n) F(-\nu\lambda_n^2/a^2)}
\end{aligned}$$

With this value of B_n we see that the expansion given by (3.9) is identical with the condition (3.8).

4. A Particular Case.

If the pressure gradient is given by

$$\frac{1}{\varrho}\frac{\partial p}{\partial x} = Ae^{-\alpha t}$$

we immediately get
$$f(s) = A\int_0^{\infty} e^{-(s+\alpha)t} dt = \frac{A}{s+\alpha}$$

The velocity is now given by the integral

$$u(y, t) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s(s+\alpha)} \frac{J_0(iy\sqrt{s/\nu}) - J_0(ia\sqrt{s/\nu})}{J_0(ia\sqrt{s/\nu})} ds$$

Thus, we get

$$u(y,\,t) = -2Aa^2\sum_{n=1}^{\infty}\frac{e^{-\nu\lambda_n^2 t/a^2}}{(\alpha a^2-\nu\lambda_n^2)}\,\frac{J_0(y/a\lambda_n)}{\lambda_n J_1(\lambda_n)}+\frac{Ae^{-\alpha t}}{\alpha}\left\{1-\frac{J_0(y(a^2\alpha/\nu)^{\frac{1}{2}}/a)}{J_0((a^2\alpha/\nu)^{\frac{1}{2}})}\right\} \tag{4.1}$$

At $t = 0$, $u(y,\,0) = 0$, therefore, we must have

$$\frac{1}{\alpha}\left\{1-\frac{J_0(yk/a)}{J_0(k)}\right\}= 2a^2\sum_{n=1}^{\infty}\frac{J_0(y\lambda_n/a)}{(\sigma a^2-\nu\lambda_n^2)\lambda_n J_1(\lambda_n)} \tag{4.2}$$

where $$k^2 = \alpha a^2/\nu$$

To prove the above equality we find out the Fourier-Bessel expansion of the left hand side of (4.2). Let

$$\frac{1}{k^2}\left\{1-\frac{J_0(yk/a)}{J_0(k)}\right\}=\sum_{n=1}^{\infty}B_nJ_0\left(\frac{y}{a}\lambda_n\right) \tag{4.3}$$

where $$B_n = \frac{2}{k^2J_1^2(\lambda_n)}\int_0^1\xi\left\{1-\frac{J_0(k\xi)}{J_0(k)}\right\}J_0(\xi\lambda_n)d\xi$$

and $$\xi = y/a.$$

Thus, $$B_n = \frac{2}{k^2J_1^2(\lambda_n)}\left[\frac{J_1(\lambda_n)}{\lambda_n}-\frac{\lambda_nJ_1(\lambda_n)}{(\lambda_n^2-k^2)}\right]$$

$$= \frac{2}{\lambda_nJ_1(\lambda_n)\,(k^2-\lambda_n^2)}$$

Substituting this value of B_n in (4.3) we find that it gives the relation (4.2). The initial condition gives us the Fourier-Bessel expansion of the left hand side of (4.2). It can easily be verified that the equality (4.2) is valid for $y = 0$ also.

5. Discussion. The velocity u, at any point of the liquid and at any instant is given by the expression (3.6). This value of u satisfies all the conditions and also is a solution of the partial differential equation (2.1) in the unsteady case.

In the particular case when u is given by (4.1), we get on putting $\alpha = 0$,

$$u(y,\,t) = \frac{a^2}{4\mu}\frac{dp}{dx}\left[8\sum_{n=1}^{\infty}\frac{J_0(y\lambda_n/a)}{\lambda_n^3J_1(\lambda_n)}e^{-\nu(\lambda_n^2t/a^2)}-\left(1-\frac{y^2}{a^2}\right)\right] \tag{5.1}$$

and $$\frac{dp}{dx} = \text{constatnt.}$$

The value of $u(y,\,t)$, given by (5.1) gives the unsteady flow in a circular pipe when the pressure gradient is constant. It is easy to see that after infinite time the flow becomes the well-known Poiseuille flow. The problem when the pressure gradient is constant has also been studied by Szymanski. A brief discussion of the solution obtained by him is given by S. Pai (1956, pp. 65-66).

Equation (2.1) has been solved by Sexl (1930) for the case when $f(t)$ is periodic. If we take $f(t) = -ke^{i\omega t}$ the solution in our terminology is given by (Pai, 1955, p. 302)

$$u(y, t) = \frac{-ik}{\omega} e^{i\omega t}\left\{1 - \frac{J_0(y(-i\omega/\nu)^{\frac{1}{2}})}{J_0(a(-i\omega/\nu)^{\frac{1}{2}})}\right\} \tag{5.2}$$

Solution (4 1) obtained by us can be written as

$$u(y. t) = I_1 + I_2,$$

where

$$I_1 = -2Aa^2 \sum_{n=1}^{\infty} \frac{e^{-\nu\lambda_n^2 t/a^2}}{(\alpha a^2 - \nu\lambda_n^2)} \; \frac{J_0(y/a\lambda_n}{\lambda_n J_1(\lambda_n)}$$

and

$$I_2 = \frac{Ae^{-\alpha t}}{\alpha}\left\{1 - \frac{J_0(y(a^2\alpha/\nu)^{\frac{1}{2}}/a)}{J_0((a^2\alpha/\nu)^{\frac{1}{2}})}\right\}$$

It is easy to see that I_1 is a general solution of the equation

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right)$$

and I_2 is the solution of equation (2.1).

If we ignore I_1 and put $A = -k$ and $\alpha = -i\omega$ our solution tallies with (5.2) with the necessary modification in the initial conditon (2 8) (*ii*).

In the case of a non-Newtonian fluid equation (2.1) remains unchanged but equation (2.2) becomes

$$0 = -\frac{\partial p}{\partial y} + \nu_0\left\{2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\left(\frac{\partial u}{\partial y}\right)^2\right\}$$

where ν and ν_0 are supposed to be constants.

Thus in a circular pipe of uniform cross-section the flow of a non-Newtonian liquid for which ν and ν_0 are constants will remain the same in character as that of a Newtonian fluid. In this case the expression for pressure will have the additional term

$$\nu_0\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \int \frac{1}{y}\left(\frac{\partial u}{\partial y}\right)^2 dy\right\}.$$

I am grateful to Dr. Ram Ballabh for the supervision and guidance in the preparation of this paper.

Summary. Assuming the pressure gradient to be a function of time, unsteady motion of a viscous homogeneous incompressible liquid contained in a circular pipe has

been studied. The problem has been solved in terms of Bessel functions. A particular case when the pressure gradient is an exponential function of time has also been studied. The results obtained are extended to non-Newtonian liquids.

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FLOW OF A NON-NEWTONIAN FLUID THROUGH AN ANNULUS WITH POROUS WALLS

By

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Introduction. The problem of steady-state laminar flow in an annulus for an ordinary viscous (Newtonian) fluid has been considered by Berman (1953). He took the fluid injection rate at one wall to be equal to the fluid suction rate at the other wall. Here we have considered the same problem under the same assumptions for a class of fluid which does not obey the general formula for Newtonian fluid, but which obeys the stress-strain relations formulated by Reiner-Braun (1952). It has been possible to obtain the solution in a closed form. As a corollary we have also deduced the expression for velocity for the laminar flow between plarallel porous plates, by making inner boundary to approach very close to the outer one. To study the effect of cross-viscosity we have taken a fixed value of the cross-flow Reynold's-no. $R (= au_a/\nu$, u_a is the radial velocity at the outer boundary) and a definite value of radius ratio σ of the walls and have considered the values of $W = w/\bar{w}$ ($\bar{w}$ being a representative velocity), for different values of K (the cross-viscosity parameter $= 2\nu_c u_a/a\nu$).

Equations. The relations between the stress and rate of strain are

$$t_{ij} = -p\delta_{ij} + t'_{ij},\ t'_{ij} = \mu e_{ij} + \mu_c e_{ik} e_{kj},$$

where μ is the viscosity coefficient of dimension $ML^{-1}T^{-1}$, and μ_c is the cross-viscosity coefficient of dimension ML^{-1}, t_{ij} is the stress tensor, e_{ij} is the rate of strain tensor and $\delta_{ij} = 1$, $i = j$ and $= 0$, $i \neq j$. In cylindrical polar coordinates

$$e_{rr} = 2\frac{\partial u}{\partial r},\ e_{\theta\theta} = \frac{2u}{r},\ e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r},\ e_{zz} = 2\frac{\partial w}{\partial z},\ e_{r\theta} = e_{\theta z} = 0,$$

where u, w are the radial and axial components of velocity respectively. From symmetry we have taken the azimuthal component to be zero. We choose the boundaries of the annulus to be $r = a$ and $r = b (a > b)$. In writing down the equations of motion we shall take $\lambda = r/a$ to represent the radial distance in a dimensionless form. We have then,

$$t_{rr} = -p + \mu e_{rr} + \mu_c(e^2_{rr} + e^2_{rz}),\ t_{\theta\theta} = -p + \mu e_{\theta\theta} + \mu_c e^2_{\theta\theta},$$

$$t_{rz} = \mu e_{rz} + \mu_c(e_{rr} + e_{zz})e_{rz},\ e_{zz} = -p + \mu e_{zz} + \mu_c(e^2_{zz} + e^2_{rz}),$$

$$t_{r\theta} = t_{\theta z} = 0.$$

The equations of motion give

$$\frac{u}{a}\frac{\partial u}{\partial \lambda} + w\frac{\partial u}{\partial z} = -\frac{1}{\varrho a}\frac{\partial p}{\partial \lambda} + \frac{1}{\varrho}\left[\frac{\partial t'_{rr}}{a\partial \lambda} + \frac{\partial t'_{rz}}{\partial z} + \frac{t'_{rr} - t'_{\theta\theta}}{a\lambda}\right], \tag{1}$$

$$\frac{u}{a}\frac{\partial w}{\partial \lambda}+w\frac{\partial w}{\partial z}=-\frac{1}{\varrho a}\frac{\partial p}{\partial z}+\frac{1}{\varrho}\left[\frac{1}{a\lambda}\frac{\partial}{\partial \lambda}(\lambda t'_{rz})+\frac{\partial t'_{zz}}{\partial z}\right], \tag{2}$$

and the equation of continuity gives

$$\frac{1}{a}\frac{\partial u}{\partial \lambda}+\frac{u}{a\lambda}+\frac{\partial w}{\partial z}=0, \tag{3}$$

ϱ being the density of the fluid. Since the injection rate at the inner wall $r = b$ has been taken as the suction rate at the outer wall $r = a$, so the axial velocity w and the radial velocity u can be taken as independent of z and we have

$$au_a = bu_b, \tag{4}$$

which with (3) leads to

$$u = \frac{u_a}{\lambda} = \sigma\frac{u_b}{\lambda}, \tag{5}$$

where σ is the radius ratio of the two boundaries and is equal to b/a.

Substituting this value of u from (5) in (1) and (2) we have

$$-\frac{u_a}{a\lambda^3}=-\frac{1}{\varrho a}\frac{\partial p}{\partial \lambda}+\nu_c\left[\frac{1}{a^2\lambda}\frac{\partial}{\partial \lambda}\left[\lambda\left(\frac{\partial w}{\partial \lambda}\right)^2\right]-\frac{16u^2}{a^5\lambda^5}\right] \tag{6}$$

$$\frac{u_a}{a}\frac{\partial w}{\partial \lambda}=-\frac{1}{\varrho}\frac{\partial p}{\partial z}+\frac{\nu}{a^2\lambda}\frac{\partial}{\partial \lambda}\left(\lambda\frac{\partial w}{\partial \lambda}\right)+\frac{2\nu_c}{a^3\lambda}\frac{\partial}{\partial \lambda}\left(\lambda\frac{\partial u}{\partial \lambda}\frac{\partial w}{\partial \lambda}\right), \tag{7}$$

where $\nu = \mu/\varrho, \quad \nu_c = \mu_c/\varrho.$

From (6) it follows that $\dfrac{\partial^2 p}{\partial \lambda \partial z} = 0$

i.e. $\dfrac{\partial p}{\partial z} = \text{constant} = \dfrac{\mu C}{a^2}$ (say).

Substituting this value of $\partial p/\partial z$ in (7) and assuming two dimensionless parameters,

$R = \dfrac{au_a}{\nu}$ = cross-flow Reynold's no. in Newtonian fluid,

$K = \dfrac{2\nu_c u_a}{a\nu}$ = a new parameter depending on the cross-viscosity coefficient μ_c,

equation (7) leads to

$$R\frac{dw}{d\lambda}=-C\lambda+\frac{d}{d\lambda}\left[\lambda\frac{dw}{d\lambda}\right]-K\frac{d}{d\lambda}\left[\frac{1}{\lambda}\frac{dw}{d\lambda}\right]. \tag{8}$$

Integrating (8) once with respect to λ,

$$\frac{dw}{d\lambda}-\frac{R\lambda}{\lambda^2-K}w=\frac{C\lambda^3}{2(\lambda^2-K)}+\frac{D\lambda}{\lambda^2-K}. \tag{9}$$

Integrating this again with respect to λ,

$$w = \frac{C}{2}\left[\frac{\lambda^2-K}{2-R}-\frac{K}{R}\right]-\frac{D}{R}+E(\lambda^2-K)^{R/2}, \quad R\neq 2, \tag{10}$$

where D, E are constants of integration. The boundary conditions are

$$w = 0, \quad \text{when} \quad \lambda = 1, \quad \text{and} \quad \lambda = \sigma. \tag{11}$$

Using (11), the equation (10) gives

$$w = -\frac{C}{2(2-R)}\left[1-\lambda^2-\frac{(1-\sigma^2)\{(1-K)^{R/2}-(\lambda^2-K)^{R/2}\}}{(1-K)^{R/2}-(\sigma^2-K)^{R/2}}\right]. \tag{12}$$

The average velocity $\bar{w}$ is given by

$$\bar{w} = \frac{2}{1-\sigma^2}\int_\sigma^1 w\lambda d\lambda$$

$$= -\frac{C}{2(2-R)(2+R)}\left[\frac{2(1+\sigma^2)+R(1-\sigma^2)-4K}{2}-\frac{R(1-\sigma^2)(1-K)^{R/2}}{(1-K)^{R/2}-(\sigma^2-K)^{R/2}}\right] \tag{13}$$

Eliminating C between (12) and (13), We express the new velocity $W = \frac{w}{\bar{w}}$ as

$$W = 2(2+R)\frac{(1-\lambda^2)\{(1-K)^{R/2}-(\sigma^2-K)^{R/2}\}-(1-\sigma^2)\{(1-K)^{R/2}-(\lambda^2-K)^{R/2}\}}{\{2(1+\sigma^2)-4K\}\{(1-K)^{R/2}-(\sigma^2-K)^{R/2}\}-R(1-\sigma^2)\{(1-K)^{R/2}+(\sigma^2-K)^{R/2}\}} \tag{14}$$

Integrating (6) with respect to λ we express the radial variation of pressure in the form

$$\frac{p(z,1)-p(z,\lambda)}{\varrho u_a^2/2} = \frac{1}{\lambda^2}-1-\frac{2K}{R}\left(\frac{1}{\lambda^2}-1\right)-\frac{K}{8R}\cdot\frac{C^3}{u_a^2}\,\frac{1}{(2-R)^2}\left[\lambda^3\left\{2-\frac{R(\lambda^2-K)^{(R-2)/2}}{(1-K)^{R/2}-(\sigma^2-K^{R/2}}\right\}^2\right.$$

$$\left.-\left\{2-\frac{R(1-K)^{(R-2)/2}}{(1-K)^{R/2}-(\sigma^2-K)^{R/2}}\right\}^2\right]. \tag{15}$$

Also integrating (7) we get the expression for the axial variation of pressure in the form

$$\frac{p(0,\lambda)-p(z,\lambda)}{\varrho\bar{w}^2/2}\cdot\frac{\bar{w}a}{8\nu}\cdot\frac{a}{z} = \frac{(4-R^2)[(1-K)^{R/2}-(\sigma^2-K)^{R/2}]}{\{2(1+\sigma^2)-4K\}\{(1-K)^{R/2}-(\sigma^2-K)^{R/2}\}-R(1-\sigma^2)\{(1-K)^{R/2}+(\sigma^2-K)^{R/2}}$$

$$= g(\sigma, R, K), \text{ say.} \tag{16}$$

When $K = 0$, expressions (14), (15) and (16) give the results for a Newtonian fluid,

$$\frac{w}{\bar{w}} = 2(2+R)\frac{(1-\lambda^2)(1-\sigma^R)-(1-\sigma^2)(1-\lambda^R)}{2(1+\sigma^2)(1-\sigma^R)-R(1-\sigma^2)(1+\sigma^R)}, \qquad \frac{p(z,1)-p(z,\lambda)}{\varrho u_a^2/2} = \frac{1}{\lambda^2}-1,$$

$$\frac{p(0,\lambda)-p(z,\lambda)}{\varrho\bar{w}^2/2}\;\frac{\bar{w}a}{8\nu}\;\frac{a}{z} = \frac{(4-R^2)(1-\sigma^R)}{2(1+\sigma^2)(1-\sigma^R)-R(1-\sigma^2)(1+\sigma^R)} = g_0(\sigma, R),$$

which are the same expressions as obtained by Berman. To deduce the expression for velocity for non-Newtonian flow through rigid wall annulus, we make $u_a \to 0$ in the equation (14) and obtain

$$\frac{w}{\overline{w}} = 2\frac{(1-\lambda^2)\ln\sigma-(1-\sigma^2)\ln\lambda}{(1+\sigma^2)\ln\sigma+(1-\sigma^2)} \tag{17}$$

From (17) it appears that the velocity profiles for a solid wall annulus will be same in both the Newtonian and non-Newtonian fluids, as was obtained by several authors. Here the maximum velocity occurs nearer the inner wall.

The velocity profile for a porous wall has the maximum pt. at $\lambda = (1+\sigma)/2$, *i.e.* midway between the two walls, when $R = 1$, $K = 0$, for any value of σ. We note that the expressions given by (14), (15) and (16) are real for all values of R, if $K \geq 1$, or $\leq \sigma^2$. This imposes a limitation on u_a in order that our assumption of steady laminar flow may be valid for a certain liquid and a certain radius ratio, *i.e.* u_a must be $\geq \frac{a\nu}{2\nu_e}$, or $\leq \frac{b^2\nu}{2a\nu_e}$.

To visualise the effect of K on the velocity profile, we give in the following table the values of W for different values of λ and K when $\sigma = 0.2$ and $R = 1$. The reason for this particular choice of R is the fact that velocity profile is symmetrical for this value of R when $K = 0$. So our table and fig. 1 will clearly show the effect of the presence of K. It is observed that when $K \leq \sigma^2$, the maximum pts. gradually shift towards the inner boundary as K increases and there is no pt. of inflexion for this value of R and σ. But when $K \geq 1$, inflexion pts. appear and these inflexion pts. and maximum pts. shift towards the inner boundary as K increases.

TABLE—1

Values of W when $\sigma = 0.2$ and $R = 1$.

$\lambda \diagdown K$	0.00	0.01	0.02	0.03	0.04	1.00	2.00
0.2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3	0.6562	0.6921	0.7445	0.8204	1.0560	0.1536	0.2162
0.4	1.1250	1.1562	1.1942	1.2456	1.3720	0.3630	0.8968
0.6	1.5000	1.5011	1.4986	1.4907	1.3250	0.8988	1.2876
0.8	1.1250	1.1097	1.0919	1.0619	0.9910	1.4 44	1.4883
0.9	0.6562	0.6438	0.6305	0.6090	0.5600	1.4814	1.0686
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Maximum values of W occur at,

$K =$	0.00	0.01	0.02	0.03	0.04	1.00	2.00
$\lambda =$	0.6000	0.5925	0.5832	0.5694	0.5280	0.8718	0.7478

Inflexion pts. of W occur at,

$K =$	1.00	2.00	3.00
$\lambda =$	0.6152	0.4549	0.4396

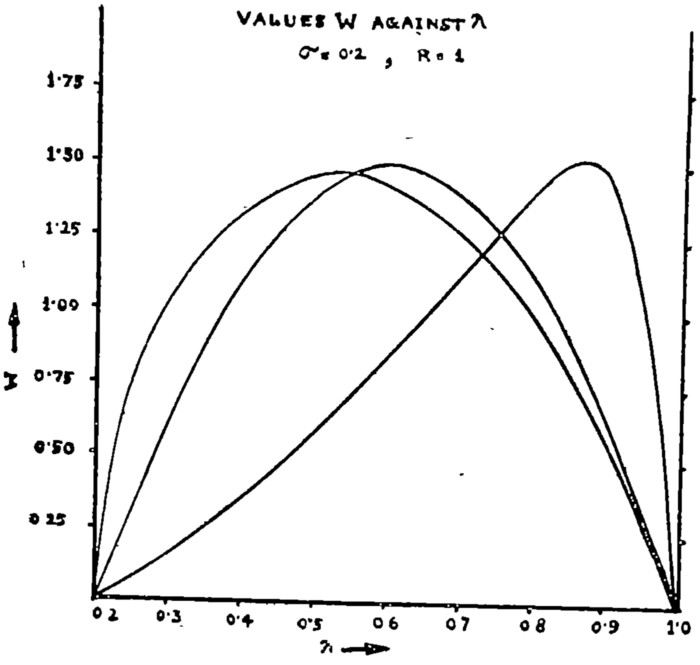

Fig. 1.

The curves correspond to the vatues of $K = 0.02, 0.00, 1.00$ respectively from left to right.

It appears from the above analysis and the expression for W, that the effect of K on the flow is the same as that of the change in the radius ratio. An increment in K affects the flow in such a way as if the radius ratio is decreased and a decrease in the value of K means an increase in the radius ratio.

Appendix.

I. *Solution of equation* (9) *when* $R = 2$. If $R = 2$, equation (9) can be written as,

$$\frac{dw}{d\lambda} - \frac{2\lambda w}{\lambda^2 - K} = \frac{C\lambda^3}{2(\lambda^2 - K)} + \frac{D\lambda}{\lambda^2 - K}. \tag{18}$$

Integrating this with respect to λ,

$$w = \frac{C}{4}\left[(\lambda^2-K)\ln(\lambda^2-K)-\frac{K}{4}\right]-\frac{D}{2}+E(\lambda^2-K). \tag{19}$$

The boundary conditions are

$$w = 0 \quad \text{when } \lambda = 1, \quad \text{and } = \sigma.$$

$$\therefore\ w = \frac{C}{4}\left[\frac{\lambda^2-K}{1-\sigma^2}\{(1-\sigma^2)\ln(\lambda^2-K)-(1-K)\ln(1-K)+(\sigma^2-K)\ln(\sigma^2-K)\}+\frac{\sigma^2-K}{1-\sigma^2}(1-K)\times\right.$$
$$\left.\times\ln\frac{1-K}{\sigma^2-K}\right] \tag{20}$$

And when $K = 0$,

$$w = \frac{c}{2(1-\sigma^2)}\left[(1-\sigma^2)\lambda^2\ln\lambda-\sigma^2(1-\lambda^2)\ln\sigma\right], \tag{21}$$

which will be the expression for w for a Newtonian fluid with Reynold's no. $=2$.

In a similar manner as in the preceding analysis we shall obtain the expressions for the axial and radial changes of pressure.

II. If the fluid is injected at the outer surface and is withdrawn at the inner boundary, u_a and u_b will be negative and so R and K will be both negative, The expression for W from (14 will be then

$$W = 2(2-R)\ \frac{(1-\lambda^2)\{(1+K)^{-R/2}-(\sigma^2+K)^{-R/2}\}-(1-\sigma^2)\{(1+K)^{-R/2}-(\lambda^2+K)^{-R/2}\}}{\{2(1+\sigma^2)+4K\}\{(1+K)^{-R/2}-(\sigma^2+K)^{-R/2}\}+R(1-\sigma^2)\{(1+K)^{-R/2}+(\sigma^2+K)^{-R/2}\}} \tag{22}$$

We can get similar expressions for (15) and (16). It appears from the above expression for W, that it is valid for all ranges of values of K, *i e.* our assumption of laminar flow is valid for all values of the cross flow u_a and the cross viscosity μ_c. In the following table we give the values of λ for which W is maximum for different values of K. Here it is observed that as K increases the maximum pts. shift towards the outer boundary, as should be expected from our previous observation. Further, if we set $d^2W/d\lambda^2 = 0$, it can be seen that inflexion pts. may appear if $K \geq R+2-\sigma^2$. So if we take $R = 1$, $\sigma = 0.2$, inflexion pts. will appear if $K \geq 2.96$.

TABLE—2.

Values of λ for which maximum values of W occur.

$K = 0.0$	0.1	0.2	0.3	0.4
$\lambda = 0.4933$	0.5719	0.6116	0.6243	0.6378.

III. The expression for velocity can also be obtained for the flow between two porous parallel plates. It we assume the injection rate at one plate to be equal to the suction rate at the other plate, then the cross flow velocity will be same at every pt.

within the plate and will be equal to the injection or suction velocity. We assume this to be u. It the distance between the plates is h, then taking x-axis perpendicular to the plates and z-axis parallel to the flow direction, we have

$$\frac{w}{\bar{w}} = 2R' \frac{\lambda(1-e^{R'})-(1-e^{R'\lambda})}{(R'-2)(1-e^{R'})-2R'}, \tag{23}$$

where

$$\lambda = \frac{x}{h}, \quad R' = \frac{uh}{\nu}.$$

This expression is obviously independent of cross-viscous coefficient, which is valid for all valus of R' and the velocity profile has got no inflexion pt., as can be verified easily.

Now this expression can also be obtained from the expression for W given by equation (14), if we make the radius ratio σ tend to one *i.e.* if the inner boundary approaches the outer boundary (both radii tending to infinity). To show this we choose the variables in the forms

$$\zeta = \frac{\lambda-\sigma}{1-\sigma}, \ \sigma = 1-\epsilon, \ R' = R(1-\sigma) \text{ and } K' = \frac{K}{1-\sigma}, \tag{24}$$

where $\epsilon \to 0$ as $\sigma \to 1$.. From (24) and (14),

$$\frac{W}{2(2\epsilon+R')} = 2R'.\frac{2(1-\zeta)\left[e^{-K'R'/2}-e^{-(2+K')R'/2}\right]-2\left[e^{-K'R'/2}-e^{-(2-2\zeta+K')R'/2}\right]}{4\left[e^{-K'R'/2}-e^{-(2+K')R'/2}\right]-2R'\left[e^{-K'R'/2}+e^{-(2+K')R'/2}\right]}$$

$$= 2R'.\frac{e^{R'}-1}{(2-R')(e^{R'}-1)-2R'}\left[\frac{e^{R'\zeta}-1}{e^{R'}-1}-\zeta\right]. \tag{25}$$

It is readily seen that expression (25) is the same as the expression (23) if we write λ for ζ.

Conclusion. (1) The flow properties depend on the parameters, σ (the radius ratio), R (cross-flow Reynold's no.), K (the dimensionless cross-viscous parameter).

(2) When the fluid is injected at the inner boundary and withdrawn at the other boundary, then our assumption of laminar flow is valid if $K \geq 1$ or $\leq \sigma^2$.

(3) For the laminar motion between porous parallel plates, the velocity does not depend on the cross-viscous coefficient μ_c. The expression for velocity and other quantities can easily be obtained from the expressions for the annular case if the inner boundary comes very close to the outer one, both radii tending to infinity.

(4) The flow through an annulus is symmetrical when $K = 0$, $R = 1$.

The velocity profile becomes more and more unsymmetrical as K increases, the maximum velocity occurring nearer the inner boundary if $K > 0$, and nearer the outer one if $K < 0$

(5) When the fluid is injected at the outer boundary and withdrawn at the inner one, our assumption for laminar flow is valid for all ranges of values of R and K.

In conclusion I express my deep gratitude to Prof. Dr. B. Sen for his guidance in pursuing this work.

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BOOK REVIEW

E. A. Maxwell—*Advanced Algebra, Part I*, Cambridge, at the University Press, 1960, 16s.

This book is written for students in upper forms of schools. The topics treated are as follows:

1 Some fundamental ideas and notations. **2** The linear polynomial. **3** The quadratic polynomial. **4** Complex numbers. **5** Transformation of equations, synthetic division and the remainder theorem. **6** Introduction to the general polynomial. **7** Solution of equations. **8** Solution of simultaneous equations. **9** The graph of a polynomial. **10** Partial fractions. **11** Inequalities. **12** The graph of a rational function. **13** Permutation and combinatiens **14** The binomial theorem. **15** The summation of series. **16** Infinite series. **17** The binomial series. **18** The exponential series. **19** The logarithmic series. **20** Elementary properties of determinants.

The author has succeeded in presenting the topics chosen logically and clearly at the level in view. His treatment of partial fraction is somewhat new. Use of the term "*palindromic equations*" (p. 75) in place of standard form of reciprocal equations is also an innovation but of no mathematical import. The book is well illustrated by worked out examples and there is a good number of examples to be worked out by the student. It is the author's "firm conviction that mathematics, like music with which it is often associated, cannot yield its full thrills to those who will not endure practice" He is right. The book is expected to be useful to those for whom it is meant.

P. K. GHOSH

E. M. Hartley—*Cartesian Geometry of the Plane*, Cambridge, at the University Press, 1960, 20s.

This is a first course in coordinate geometry of the plane meant for students working for the Advanced and Scholarship levels of the General Certificate of Educatian (Great Britain) and it is expected meet partially the needs of those reading for a General degree including Mathematics. The author claims that he has tried to combine clarity with more rigour than seems to be usual at this stage. In reviewer's opinion the author has still left much to be desired as regards rigour even at this stage. The use of calculus to obtain equations to the tangents to conics even in their simplest forms requires that certain special cases are to be treated separately. But the author has paid no attention to these points. Apart from these blemishes, the author has succeeded as regards his claim for clarity. The contents of the book are

I The point. **II** The line. **III** The parabola. **IV** Curves defined parametrically: the Rectangular Hyperbola, Cubic Curves, Trigonometric curves **V** The circle. **VI** The Ellipse and the Hyperbola **VII** Line pairs. **VIII** The General conic. **XI** Polar coordinates. **X** What is a conic? There is also an introduction containing topics from Algebra, Pure Geometry, Trigonometry and Calculus which are used in the book.

The book contains a fair number of worked out illustrative examples and also a good number of examples to be worked out by the student.

P. K. GHOSH

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ON AXIALLY-SYMMETRIC NON-NEWTONIAN FLOWS

By

M. M. Oberoi, *Paris* and J. N. Kapur, *Kanpur*

(*Received—March 5, 1960*)

1. **Introduction.** Non-Newtonian flows have been discussed by Reiner (1945, 1948), Rivlin (1948*a*,*b*, 1949) Oldroyd (1958), and others. Certain axially-symmetric problems *e.g.* flow through a circular tube [Rivlin (1949); Kapur; Rivlin & Green (1956); Bhatnagar and Rao (1958); flow in the annulus between two circular cylinders [Serrin (1959) Rathna (1959); Narasimhan (1959,], rotation of a disc [Srivastava (1958); Mittal (1958)], oscillation of a disc [Srivastava, 1960] etc. have also been discussed in the literature. In this paper it is proposed to give a systematic discussion of axially-symmetric non-Newtonian fluid flows and derive some new solutions for this case. For this purpose, we use the method given in Goldstein (1938) and recently used extensively by Chandrasekhar (1956*a*, *b*) and Kapur (1958, 1959*a*,*b*) for discussing axially-symmetric viscous or axially-symmetric hydromagnetic flows. The method consists in separating out the poloidal and toroidal components of the velocity field in terms of two functions ψ and Ω and choosing such a form for the velocity vector that the equation of continuity is automatically satisfied.

We first write down the equations for axially-symmetric non-Newtonian flows and then discuss the case when ψ and Ω are functions of r only so that the velocity has no radial component and the axial and toroidal components of the velocity are functions of r only. In this case we find that the coefficient of ν_c, the coefficient of kinematic cross-viscosity vanishes in the equations so that the velocity field is the same for Newtonian and non-Newtonian flows. We obtain solutions for steady flows of this type and show that the solutions remain valid even if the variability of μ_c is taken into account, but only for a very special class of irrotational flows, the solution is valid if the variability of μ is also taken into account.

We also consider the axially-symmetric non-Newtonian flows in which all the components of the velocity are functions of r only and show that it includes the case of non-Newtonian flow in an annulus between two circular cylinders with suction and injection as a particular case and the result in this case shows, in a very interesting manner, the effect of cross-viscosity.

2. Equations of Motion. In cylindrical polar coordinates r, θ, z with the axis of symmetry in the direction of the z-axis, the equations of motion of axially-symmetric case *i.e.* when $\partial/\partial\theta \equiv 0$ are:

$$\varrho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)=\frac{\partial t_{rr}}{\partial r}+\frac{\partial t_{zr}}{\partial z}+\frac{t_{rr}-t_{\theta\theta}}{r} \tag{1}$$

$$\varrho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}\right)=\frac{\partial t_{r\theta}}{\partial r}+\frac{\partial t_{\theta z}}{\partial z}+\frac{2t_{r\theta}}{r} \tag{2}$$

$$\varrho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=\frac{\partial t_{rz}}{\partial r}+\frac{\partial t_{zz}}{\partial z}+\frac{t_{rz}}{r} \tag{3}$$

where the velocity vector $\vec{q}$ is given by

$$\vec{q}=u\vec{i}_r+v\vec{i}_\theta+w\vec{i}_z. \tag{4}$$

where $\vec{i}_r$, $\vec{i}_\theta$, $\vec{i}_z$ are the unit vectors and for non-Newtonian liquids, the stress tensor is given by

$$t_{ij}=-p\delta_{ij}+\mu d_{ij}+\mu_c D_{ij}. \tag{5}$$

Here p is the pressure, μ is the coeffecient of viscosity, μ_c is the coeffecient of cross-viscosity, d_{ij} is the rate-of-strain tensor defined by

$$d_{ij}=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i} \tag{6}$$

and

$$D_{ij}=d_{i\alpha}d_{\alpha j}, \tag{7}$$

where in the last equation and in (8) below summation convention is used. μ and μ_c are, for general, non-Newtonian liquids, functions of the three scalar invariants.

$$I_1=d_{ii},\ I_2=d_{ij}d_{ji},\ I_3=\det.\ d_{ij} \tag{8}$$

For the case of axi-symmetry.

$$[d_{ij}]=\begin{bmatrix} 2u_r & v_r-\dfrac{v}{r} & u_z+w_r \\ v_r-\dfrac{v}{r} & \dfrac{2u}{r} & v_z \\ u_z+w_r & v_z & 2w_z \end{bmatrix} \tag{9}$$

For incompressible fluids, the equation of continuity is $d_{ii}=0$, or

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{10}$$

so that in this case μ and μ_c are functions of I_2 and I_3 only. In most of our discussions, however, we shall discuss only the special class of non-Newtonian flows for which μ and μ_c are constants.

3. Equation of motion in terms of ψ and Ω. Let

$$\vec{q} = -\frac{1}{r}\frac{\partial\psi}{\partial z}\vec{i}_r + \frac{1}{r}\Omega\vec{i}_\theta + \frac{1}{r}\frac{\partial\psi}{\partial r}\vec{i}_z \tag{11}$$

then the equation of continuity (10) is automatically satisfied and (9) gives

$$[d_{ij}] = \begin{bmatrix} 2\left(\frac{\psi_z}{r^2} - \frac{\psi_{rz}}{r}\right) & \frac{\Omega_r}{r} - \frac{2\Omega}{r^2} & -\frac{1}{r}\psi_{zz} + \frac{1}{r}\psi_{rr} - \frac{\psi_r}{r^2} \\ \frac{\Omega_r}{r} - \frac{2\Omega}{r^2} & -\frac{2\psi_z}{r^2} & \frac{\Omega_z}{r} \\ -\frac{1}{r}\psi_{zz} + \frac{\psi_{rr}}{r} - \frac{\psi_z}{r^2} & \frac{\Omega_z}{r} & \frac{2\psi_{rz}}{r} \end{bmatrix} \tag{12}$$

From this we can compute the elements of the matrix $[D_{ij}]$ and substitute in equations (1), (2) and (3).

Equation (2) gives

$$\Omega_t + \frac{1}{r}\frac{\partial(\psi, \Omega)}{\partial(r, z)} = \nu D^2\Omega + \nu_c r\left[\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)D_{12} + \frac{\partial}{\partial z}D_{23}\right] \tag{13}$$

where
$$\nu = \frac{\mu}{\varrho}, \qquad \nu_c = \frac{\mu_c}{\varrho},$$

and
$$D^2 \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{14}$$

Again from (1) and (3), eliminating p, we get

$$\frac{\partial}{\partial t}(D^2\psi) + \frac{2\Omega\Omega_z}{r^2} + \frac{1}{r}\frac{\partial(\psi, D^2\psi)}{\partial(r, z)} - \frac{2}{r^2}\psi_z D^2\psi$$

$$= \nu D^4\psi + r\nu_c\left\{\begin{array}{c} \frac{\partial^2}{\partial r^2}(D_{13}) + \frac{\partial}{\partial r}\left(\frac{D_{13}}{r}\right) + \frac{\partial^2}{\partial r\partial z}(D_{33} - D_{11}) \\ -\frac{\partial}{\partial z}\left(\frac{D_{11} - D_{22}}{r}\right) - \frac{\partial^2}{\partial z^2}D_{13} \end{array}\right\} \tag{15}$$

Equations (13) and (15) give two equations to determine ψ and Ω. If $\nu_c = 0$, they reduce to the corresponding equations for ordinary viscous fluids.

4. Special cases. *Case 1. Purely poloidal velocity fields*: In this case $\Omega = 0$, and (12) gixes

$$[d_{ij}] = \begin{bmatrix} 2\left(\dfrac{\psi_z}{r^2} - \dfrac{\psi_{rz}}{r}\right) & 0 & -\dfrac{1}{r}\psi_{zz} + \dfrac{1}{r}\psi_{rr} - \dfrac{\psi_r}{r^2} \\ 0 & -\dfrac{2\psi_z}{r^2} & 0 \\ -\dfrac{1}{r}\psi_{zz} + \dfrac{1}{r}\psi_{rr} - \dfrac{\psi_r}{r^2} & 0 & \dfrac{2\psi_{rz}}{r} \end{bmatrix} \tag{16}$$

Equation (13) is identially satisfied and equation (15) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}D^2\psi + \frac{1}{r}\,\frac{\partial(\psi, D^2\psi)}{\partial(r, z)} + \frac{2}{r}\psi_z D^2\psi \\ &= \nu D^4\psi + r\nu_c \left\{ \begin{array}{l} \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} - \dfrac{\partial^2}{\partial z^2}\right)\left[\dfrac{2\psi_z}{r^3}\left(\psi_{rr} - \psi_{zz} - \dfrac{\psi_r}{r}\right)\right] \\ \qquad\qquad + 4\dfrac{\partial^2}{\partial r \partial z}\left[\dfrac{2\psi_z\psi_{rz}}{r^3} - \dfrac{\psi_z^2}{r^4}\right] \\ -\dfrac{\partial}{\partial_z}\,\dfrac{1}{r}\left[4\left(\dfrac{\psi_z}{r^2} - \dfrac{\psi_{rz}}{r}\right)^2 + \left(\dfrac{\psi_{rr}}{r} - \dfrac{\psi_{zz}}{r} - \dfrac{\psi_r}{r^2}\right)^2 - \dfrac{4\psi_z^2}{r^4}\right] \end{array} \right\} \end{aligned} \tag{17}$$

Case II. Ω and ψ are functions of r only. In this case $u = 0$, and

$$[d_{ij}] = \begin{bmatrix} 0 & \dfrac{\Omega_r}{r} - \dfrac{2\Omega}{r^2} & \dfrac{1}{r}\psi_{rr} - \dfrac{1}{r^2}\psi_r \\ \dfrac{\Omega_r}{r} - \dfrac{2\Omega}{r^2} & 0 & 0 \\ \dfrac{\psi_{rr}}{r} - \dfrac{\psi_r}{r^2} & 0 & 0 \end{bmatrix} \tag{18}$$

$$[D_{ij}] = \begin{bmatrix} \left(\dfrac{\Omega_r}{r} - \dfrac{2\Omega}{r^2}\right)^2 + \left(\dfrac{1}{r}\psi_{rr} - \dfrac{1}{r^2}\psi r\right)^2 & 0 & 0 \\ 0 & \left(\dfrac{\Omega_r}{r} - \dfrac{2\Omega}{r^2}\right)^2 & \left(\dfrac{\Omega_r}{r} - \dfrac{2\Omega}{r^2}\right)\left(\dfrac{\psi_{rr}}{r} - \dfrac{\psi_r}{r^2}\right) \\ 0 & \left(\dfrac{\Omega_r}{r} - \dfrac{2\Omega}{r^2}\right)\left(\dfrac{\psi_{rr}}{r} - \dfrac{\psi_r}{r^2}\right) & \left(\dfrac{\psi_{rr}}{r} - \dfrac{\psi_r}{r^2}\right)^2 \end{bmatrix} \tag{19}$$

The equations (13) and (15) become

$$\Omega_t = \nu\left(\Omega_{rr} - \frac{\Omega_r}{r}\right) \tag{25}$$

$$\psi_{rrt} - \frac{1}{r}\psi_{rt} = \nu\left(\psi_{rrrr} - \frac{2}{r}\psi_{,rr} + \frac{3}{r^2}\psi_{rr} - \frac{3}{r^3}\psi_r\right). \tag{21}$$

We observe that in this case

(i) equations for Ω and ψ have been separated, so that the poloidal and toroidal components of the velocity field are determined independently.

(ii) Coeffecient of ν_c is zero in both the equations *i e.* the flows which can occur for ordinary Newtonian fluids are also possible for non-Newtonian fluids in this case.

For steady flows, (20) and (21) become

$$\Omega_{,r} - \frac{\Omega_r}{r} = 0 \tag{22}$$

$$r^4\psi_{rrrr} - 2r^3\psi_{rrr} + 3r^2\psi_{rr} - 3_r\psi_r = 0. \tag{22}$$

Solving these equations, we get

$$\Omega = Ar^2 + B \tag{24}$$

$$\psi = Dr + Er\log r + Fr^3, \tag{25}$$

where A, B, D. E, F are arbitrary constants. Thus

$$\vec{q} = \left(Ar + \frac{B}{r}\right)\vec{i_\varphi} + [D + E\log r + Fr^2]\vec{i_z} \tag{26}$$

gives the most general axially-symmetric flow possible in the case when ψ and Ω are functions of r only and μ and μ_c are constants.

Thus for flow of such a fluid in a circular tube, from the conditions that the velocity vanishes at the surface $r = a$ and finite on the axis *i.e.* at $r = 0$, we get

$$\vec{q} = k(a^2 - r^2)\vec{i_z} \tag{27}$$

where k is a constant.

Using the equations of motion (1), (2), (3), we get

$$\frac{\partial p}{\partial z} = -2\mu k = -P, \quad \frac{\partial p}{\partial r} = 12\mu_c k^2 r \tag{28}$$

so that
$$p = 6\mu_c k^2 r^2 - 2\mu z + C. \tag{29}$$

Thus, unlike Newtonian fluids, in the case of non-Newtonian fluids, the pressure gradient in the direction of the radius is non-zero.

Similarly for flow in the annulus between two circular cylinders, of which the inner of radius a has a velocity w in the direction of the axis and the outer of radius b is at rest, we have

$$\vec{q} = w\frac{\log(r/b)}{\log(a/b)} - \frac{P}{4\mu}\left[r^2 - \frac{b^2\log(r/a) - a^2\log(r/b)}{\log(b/a)}\right] \tag{30}$$

and $$\frac{\partial p}{\partial r} = \tfrac{1}{2}\mu_c\left\{\frac{w}{\log(a/b)}\frac{1}{r} - \frac{P}{4\mu}\left(2r - \frac{b^2-a^2}{\log(b/a)}\frac{1}{r}\right)\right\} \times \left\{\frac{w}{r^2}\frac{1}{\log(b/a)} - \frac{3P}{2\mu} - \frac{P}{4\mu}\frac{1}{\log(b/a)}\frac{b^2-a^2}{r^4}\right\} \tag{31}$$

giving the effect of cross-viscosity on the pressure gradient,

Case III. All velocity components are functions of r only,. This requires that ψ_r, ψ_z and Ω are functions of r only, so that

$$\frac{\partial}{\partial r}(\psi_z) = \frac{\partial}{\partial z}(\psi_r) = 0,$$

and ψ_r is just a constant, say K.

For steady flow (13) and (15) give

$$\frac{1}{r}\left[-K\frac{d\Omega}{dr}\right] = \nu\left[\frac{d^2\Omega}{dr^2} - \frac{1}{r}\frac{d\Omega}{dr}\right] \tag{32}$$

$$\frac{1}{r}\left[-K\frac{d}{dr}\left(D^2\psi\right)\right] + \frac{2}{r^2}KD^2\psi \tag{32}$$

$$= \nu D^4\psi + r\nu_c\left\{\frac{d^2}{dr^2}\left(\frac{2K}{r^3}D^2\psi\right) + \frac{d}{dr}\left(\frac{2K}{r^4}D^4\psi\right)\right\}. \tag{33}$$

The solutions of these two equations are :

$$\Omega = Ar^{2-R} + B \tag{34}$$

and $$D^2\psi = \frac{Cr^2}{R-2}\left[r^2+c^2\right]^{\frac{1}{2}R-1} + Dr^2, \tag{35}$$

where $$R = \frac{K}{\nu}, c^2 = \frac{2\nu_c K}{\nu} = \frac{2\nu_c}{R}. \tag{36}$$

Integrating (35), we get

$$\frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{C}{R(R-2)}(r^2+c^2)^{R/2} + \frac{Dr^2}{2} + E, \tag{37}$$

so that in this case the most general form of the velocity vector is

$$\vec{q} = -\frac{K}{r}\vec{i}_r + \left(Ar^{1-R} + \frac{B}{r}\right)\vec{i}_\theta + \left[\frac{C}{R(R-2)}(r^2+c^2)^{R/2} + \frac{Dr^2}{2} + E\right]\vec{i}_z. \tag{38}$$

As an example, consider the flow of a non-Newtonian fluid in an annulus between two circular cylinders with equal suction and injection at the two boundaries.

In this case $A = 0$, $B = 0$ and the axial component of velocity is given by

$$w = \frac{P}{R(R-2)}\left\{(r^2+c^2)^{R/2}-(a^2+c^2)^{R/2}\frac{r^2-b^2}{a^2-b^2}+(b^2+c^2)^{R/2}\frac{r^2-a^2}{a^2-b^2}\right\}. \tag{39}$$

Let us represent the corresponding axial velocity component when the crossviscosity is absent, so that

$$w_0 = \frac{P}{R(R-2)}\left\{r^R-a^R\frac{r^2-b^2}{a^2-b^2}+b^R\frac{r^2-a^2}{a^2-b^2}\right\} \tag{40}$$

which agrees with the expression obtained by Berman (1958).

If we write

$$r^2+c^2 = r'^2, \quad a^2+c^2 = a'^2, \quad b^2+c^2 = b'^2, \tag{41}$$

(39) becomes

$$w = \frac{P}{R(R-2)}\left\{r'^R-a'^R\frac{r'^2-b'^2}{a'^2-b'^2}+b'^R\frac{r'^2-a'^2}{a'^2-b'^2}\right\}. \tag{42}$$

Comparing (40) and (42) we get the most interesting result that as far as the axial velocity component is concerned its effect is as if all radial distances have changed in the ratio $(1+c^2/r^2)^{\frac{1}{2}}:1$. In other words the axial component of velocity at a distance r from the axis for non-Newtonian flow with equal suction and injection in an annulus bounded by cylinders of radii a and b is the same as the axial component at a distance $(r^2+c^2)^{\frac{1}{2}}$ for Newtonian flow in an annulus bounded by cylinders of radii $(a^2+c^2)^{\frac{1}{2}}$ and $(b^2+c^2)^{\frac{1}{2}}$.

It is easily verified that c which depends on the ratio of cross-viscosity to viscosity has the dimensions of length. Using the above result all the results which Narasimhan (1959) has deduced about the behaviour of non-Newtonian flows in an annulus can be deduced as corollaries of the corresponding results of Berman (1958).

5. The case when μ and μ_c are variable. In this case the right-hand sides of (1), (2), (3) give the following additional terms as compared with the case where μ and μ_c are constant.

$$\left.\begin{aligned}
&d_{rr}\frac{\partial\mu}{\partial r}+d_{zr}\frac{\partial\mu}{\partial z}+D_{rr}\frac{\partial\mu_c}{\partial r}+D_{zr}\frac{\partial\mu_c}{\partial z}\\
&d_{r\theta}\frac{\partial\mu}{\partial\theta}+d_{z\theta}\frac{\partial\mu}{\partial z}+D_{r\theta}\frac{\partial\mu_c}{\partial r}+D_{z\theta}\frac{\partial\mu_c}{\partial z}\\
&d_{zr}\frac{\partial\mu}{\partial r}+d_{zz}\frac{\partial\mu}{\partial z}+D_{zr}\frac{\partial\mu_c}{\partial r}+D_{zz}\frac{\partial\mu_c}{\partial z}
\end{aligned}\right\}. \tag{43}$$

We consider the case when ψ and Ω are function of r alone. In this case the invariants I_2 and I_3 are also functions of r alone and therefore μ and μ_c are also functions of r alone so that in each of the above three expressions, the second and fourth terms vanish.

Also from (18) and (19)

$$d_{rr} = 0, \quad D_{r\theta} = 0, \quad D_{rz} = 0 \tag{44}$$

so that the above three expressions reduce respectively to

$$D_{rr}\frac{\partial\mu_c}{\partial r}, \quad d_{r\theta}\frac{\partial\mu}{\partial r}, \quad d_{zr}\frac{\partial\mu}{\partial r} \tag{45}$$

The contributions of these to the right hand sides of (13) and (15) are respectively

$$\left(\frac{\Omega_r}{r} - \frac{2\Omega}{r^2}\right)\frac{\partial\mu}{\partial r} \tag{45a}$$

and

$$-\frac{\partial}{\partial r}\left[\left(\frac{1}{r}\psi_{rr} - \frac{1}{r^2}\psi_r\right)\frac{\partial\mu}{\partial r}\right]. \tag{45b}$$

Thus we deduce that the solution (26) holds not only for non-Newtonian flows with constant values of μ and μ_c, but also for these fluids for which μ is constant, but μ_c may be variable.

If μ and μ_c are both variable, then the solution which is possible is given by (26) when

$$B = E = F = 0,$$

so that

$$\vec{q} = Ar\vec{i}_\varphi + D\vec{i}_z$$

in which case the stress tensor reduces to the isotropic pressure tensor.

The above results are consistent with those obtained by Kapur (1958), where he showed the some general results for two dimensional flows of non-Newtonian fluid proved earlier by Bhatnagar and Lakshman Rao (1957), for constant values of μ and μ_c remain valid if variability of μ_c is taken into account, but they are no longer valid if the variability of μ is also taken into account.

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NOTE ON THE REMAINDER TERM IN A CERTAIN TYPE OF QUADRATURE FORMULAE

By

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It is interesting to study from the theoretical standpoint the remainder term in a closed form in quadrature formulae of the type of Weddle's rule, *viz* , formulae of Hardy, Shovelton, etc. (*cf.* Whittaker and Robinson, 1937, p. 151). For Weddle's rule and Hardy's formula the remainders in a closed from involving derivatives of *different* orders at *two different undetermined* points are known (Steffenson, 1927, p. 168). Here we shall obtain a general expression (closed) for the remainder terms in term of derivatives of the same order at *two different undetermined* points.

We shall consider a quadrature formula (symmetrical) with odd number of ordinates:

$$I \equiv \int_{x_0-nh}^{x_0+nh} f(x)dx$$

$$\simeq h[A_n f(x_0-nh)+A_{n-1}f(x_0-\overline{n-1}h)+\ldots\ldots+A_1 f(x_0-h)+A_0 f(x_0)$$

$$+A_1 f(x_0+h)+\ldots\ldots\ldots\ldots+A_{n-1}f(x_0+\overline{n-1}h)+A_n f(x_0+nh)], \qquad (1)$$

$$(A_r \geq 0, \quad r=1,2\ldots\ldots,n).$$

We shall look upon the remainder as a function of h and write this as

$$\varphi(h) \equiv F(x_0+nh)-F(x_0-nh)-h[A_n f(x_0-nh)+A_{n-1}f(x_0-\overline{n-1}h)+\ldots$$

$$\ldots+A_1 f(x_0-h)+A_0 f(x_0)+A_1 f(x_0+h)+\ \ldots+A_{n-1}f(x_0+\overline{n-1}h)$$

$$+A_n f(x_0+nh)] \qquad (2)$$

where $F'(x)=f(x)$ [*cf.* De la Vallée Poussin, 1938, p. 339].

Evidently $\varphi(h)$ is an odd function of h. We assume that $f(x_0 \pm rh)$ can be expanded by Taylor's theorem upto a sufficient number of terms. We see that $\varphi^{(2m)}(0)=0$ when it exists. Let us assume further that

$$\varphi'(0)=\varphi^{(3)}(0)=\ldots=\varphi^{(2m+1)}(0)=0 \text{ but } \varphi^{(2m+3)}(0)\neq 0 \ (m \leqq n).$$

That is, the quadrature formula is such that it integrates a polynomial of degree

$(2m+1)$ exactly. Differentiating (2) $(2m+1)$ times in succession we get

$$\begin{aligned}\varphi'(h) &= nf(x_0+nh)+nf(x_0-nh)-h[A_n n\{f'(x_0+nh)-f'(x_0-nh)\}.\\ &+A_{n-1}(n-1)\,\{f'(x_0+\overline{n-1}h)-f'(x_0-\overline{n-1}h)\}+\ldots+A_1\{f'(x_0+h)-f'(x_0-h)\}\\ &-[A_n\{f(x_0+nh)+f(x_0-nh)\}+A_{n-1}\{f(x_0+\overline{n-1}h)+f(x_0-\overline{n-1}h)\}+\ldots\\ &\ldots+A_1\{f(x_0+h)+f(x_0-h)\}+A_0 f(x_0)],\end{aligned} \tag{3}$$

$$\begin{aligned}\varphi^{(2r+1)}(h) &= n^{2r+1}[f^{(2r)}(x_0+nh)+f^{(2r)}(x_0-nh)]\\ &-h[A_n n^{2r+1}\{f^{(2r+1)}(x_0+nh)-f^{(2r+1)}(x_0-nh)\}\\ &\quad+A_{n-1}(n-1)^{2r+1}\{f^{(2r+1)}(x_0+\overline{n-1}h)-f^{(2r+1)}(x_0-\overline{n-1}h)\}\\ &\quad+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ &\quad+A_1\{f^{(2r+1)}(x_0+h)-f^{(2r+1)}(x_0-h)\}]\\ &-(2r+1)[A_n n^{2r}\{f^{(2r)}(x_0+nh)+f^{(2r)}(x_0-nh)\}\\ &\quad+A_{n-1}(n-1)^{2r}\{f^{(2r)}(x_0+\overline{n-1}h)+f^{(2r)}(x_0-\overline{n-1}h)\}+\ldots\\ &\quad+A_1\{f^{(2r)}(x_0+h)+f^{(2r)}(x_0-h)\}],\quad (r\geqq 1)\end{aligned} \tag{4}$$

As $\varphi^{(2r+1)}(0) = 0\ (0\leq r\leq m)$, we have

$$\left.\begin{aligned}2n &= 2(A_n+A_{n-1}+\ldots\ldots+A_1)+A_0\,;\\ n^{2r+1} &= (2r+1)[A_n n^{2r}+A_{n-1}(n-1)^{2r}+\ldots\ldots+A_1]\end{aligned}\right\} \tag{5}$$

Now

$$\begin{aligned}T_1 \equiv h[A_n n^{2m+1}&\{f^{(2m+1)}(x_0+nh)-f^{(2m+1)}(x_0-nh)\}\\ &+A_{n-1}(n-1)^{2m+1}\{f^{(2m+1)}(x_0+\overline{n-1}h)-f^{(2m+1)}(x_0-\overline{n-1}h)\\ &+\ldots\ldots+A_1\{f^{(2m+1)}(x_0+h)-f^{(2m+1)}(x_0-h)\}\end{aligned}$$

$$\begin{aligned}&= 2h^2[A_n n^{2m+2}f^{(2m+2)}(\xi_n)+A_{n-1}(n-1)^{2m+1}f^{(2m+2)}(\xi_{n-1})+\ldots\ldots+A_1 f^{(2m+2)}(\xi_1)]\\ &= 2h^2[A_n n^{2m+2}+A_{n-1}(n-1)^{2m+2}+\ldots\ldots+A_1]f^{(2m+2)}(\mu_1) = P_1 h^2 f^{(2m+2)}(\mu_1)\end{aligned} \tag{6}$$

where $P_1 = 2[A_n n^{2m+2}+A_{n-1}(n-1)^{2m+2}+\ldots\ldots+A_1]$ (7)

and $\xi_n,\xi_{n-1},\ldots\xi_1,\mu_1$ all lie between x_0-nh and x_0+nh;

$$\begin{aligned}T_2 \equiv n^{2m+1}&\{f^{(2m)}(x_0+nh)+f^{(2m)}(x_0-nh)\}\\ &-(2m+1)[A_n n^{2m}\{f^{(2m)}(x_0+nh)+f^{(2m)}(x_0-nh)\}\\ &\quad+A_{n-1}(n-1)^{2m}\{f^{(2m)}(x_0+\overline{n+1}h)+f^{(2m)}(x_0-\overline{n-1}h)\}\\ &\quad+\ldots\ldots\\ &\quad+A_1\{f^{(2m)}(x_0+h)+f^{(2m)}(x_0-h)\}]\\ &= n^{2m+2}[n-(2m+1)A_n]h^2 f^{(2m+2)}(\zeta_n)\\ &\quad-(2m+1)h^2\{A_{n-1}(n-1)^{2m+2}f^{(2m+2)}(\zeta_{n-1})+\ldots\ldots+A_1 f^{(2m+2)}(\zeta_1)\}\\ &= n^{2m+2}[n-(2m+1)A_n]h^2 f^{(2m+2)}(\mu_2)\\ &\quad-(2m+1)[A_{n-1}(n-1)^{2m+2}+\ldots\ldots A_1]f^{(2m+2)}(\mu_3)\end{aligned}$$

$$= P_2h^2f(\mu_2) - P_3h^2f^{(2m+2)}(\mu_3). \tag{8}$$

where
$$\left.\begin{aligned} P_2 &= n^{2m+2}[n-(2m+1)A_n] \\ P_3 &= (2m+1)[A_{n-1}(n-1)^{2m+2}+\ldots+A_1] \end{aligned}\right\} \tag{9}$$

and $\zeta_n, \zeta_{n-1}, \ldots, \zeta_1 \cdot \mu_2, \mu_3$ between x_0-nh and x_0+nh.

Thus $$\varphi^{(2m+1)}(h) = -P_1h^2f^{(2m+2)}(\mu_1)+P_2h^2f^{(2m+2)}(\mu_2)-P_3h^2f^{(2m+2)}(\mu_3) \tag{10}$$

But $P_1 \geq 0$, $P_3 \geq 0$ essentially and $P_2 > 0$ for the cases under consideration so that we may write

$$\varphi^{(2m+1)}(h) = -C_1h^2f^{(2m+2)}(\eta'_1)+C_2h^2f^{(2m+2)}(\mu_2) \tag{11}$$

where $C_1 = P_1+P_2$, $C_1 = P_2$ and η'_1 lies between x_0-nh and x_0+nh.

Integrating equation (11) with respect to h from 0 to h $(2m+1)$ times in succession and using the mean-value theorem, we get

$$\varphi(h) = -2C_1\frac{h^{2m+3}}{(2m+3)!}f^{(2m+2)}(\eta)+2C_2\frac{h^{2m+3}}{(2m+3)!}f^{(2m+2)}(\eta') \tag{12}$$

where η, η' lie between x_0-nh and x_0+nh.

Now for Simpson's one-third rule we have, taking two intervals only, $n = 1$, $m = 1$, $A_1 = \frac{1}{3}$, $A_0 = \frac{4}{3}$, so that $P_1 = \frac{2}{3}$, $P_2 = 0 = P_3$ in (10) and we have the remainder term

$$\varphi(h) = -\tfrac{4}{3}\cdot\frac{h^6}{5!}f^{(4)}(\eta) \tag{13}$$

For Weddle's rule (with 6 intervals) we have $n = 3$, $m = 2$;
$A_3 = \frac{3}{10}$, $A_2 = \frac{3}{10}\times 5$, $A_1 = \frac{3}{10}$, $A_0 = \frac{5}{10}.6$; $P_1 = 630$, $P_2 = 1093\frac{1}{2}$, $P_3 = 481\frac{1}{2}$;

Thus $$\varphi(h) = -2223\frac{h^7}{7!}f^{(6)}(\eta)+2187\frac{h^7}{7!}f^{(6)}(\eta'), \tag{14}$$

a well-known result.

We observe that this expression for the remainder involves the values of $f^{(6)}(x)$ at two undetermined points within the interval of integration. If we can neglect derivatives of $f(x)$ higher than the sixth *i.e.* consider the sixth derivative as practically constant, then

$$\varphi(h) \simeq -\tfrac{1}{140}h^7f^{(6)}(x_0), \tag{15}$$

a well-known result.

We can also write in place of equations (6) and (8)

$$T_1 = P_1h^2f^{(2m+2)}(x_0)+Q_1h^4f^{(2m+4)}(\mu'_1) \tag{6$'$}$$

where $$Q_1 = \frac{4}{3!}[A_nn^{2m+4}+A_{n-1}(n-1)^{2m+4}+\ldots+A_1] \tag{7}$$

and $$T_2 = (P_2-P_3)h^2f^{(2m+2)}(x_0)+Q_2h^4f^{(2m+4)}(\mu'_2)-Q_3h^4f^{(2m+4)}(\mu'_3) \tag{8}$$

where Q_2 and Q_3 are experessions involving A's, n and m

and μ'_1, μ'_2, and μ'_3 lie between x_0+nh and x_0+nh

Here also, $Q_1 \geq 0$, $Q_3 \geq 0$ essentially and $Q_2 > 0$ for formulae under consideration. Proceeding as above

$$\varphi(h) = (P_2 - P_3 - P_1)\frac{2h^7}{7!}f^{(2m+2)}(x_0) - D_1\frac{3!h^9}{9!}f^{(2m+4)}(\mu'_1) + D_2\frac{3!h^9}{9!}f^{(2m+4)}(\mu'_2) \qquad (16)$$

where η'_1 and η'_2 lie in the interval x_0-nh and x_0+nh, and D', D_2 are positive quantities.

This exprsssion for the remainder involves the values of the $(2m+4)$th derivative of $f(x)$ at two undertermined points. In case this derivative is negligible

$$\varphi(h) \simeq (P_3 - P_3 - \mathrm{P}_1)\frac{2h^7}{7!}f^{(2m+2)}(x_0) \qquad (17)$$

and in particular for Weddle's rule (with 6 intervals) this reduces

$$\varphi(h) \simeq -\frac{h^7}{140}.f^{(2m+2)}(x_0)$$

which is the same as (15).

For Hardy's rule, Shovelton's rule, etc. the remainder can be estimated by substituting the values of the A's, n and m in the expressions for P's and Q's.

Analogons expressions for the remainder may be found when the number of ordinates involved is even.

It should be noted that in the case Newton-Cotes formulae expressions for the remainders are known which involve the value of a derivative at one undetermined point.

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NOTE ON THE BENDING OF A THIN UNIFORMLY COMPRESSED CIRCULAR PLATE BY CONECNTRATED AND NON-UNIFORM LOAD. PART—I

By

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Introduction. Though the solution of the problem of circular plates submitted to the simultaneous action of uniform compression in the middle plane of the plate and uniformly distributed transverse load is known (Timoshenko & Woinowsky Krieger, 1959) it is believed that the corresponding solutions for a concentrated load or any other type of lateral load have not been obtained. In this note, the deflection due to a concentric concentrated load in a uniformly compressed circular plate with a clamped edge is first solved, and then a general solution for loads which are functions of the distance from the centre is discussed.

1. Problem. Deflection w of an elastic plate under uniform compression satisfies the equation

$$D\nabla_1^4 w + T\nabla_1^2 w = Z$$

Where D is the flexural rigidity of the plate

$$= \tfrac{2}{3}\frac{Eh^3}{(1-\sigma^2)}$$

T = normal compressive force per unit length of the boundary,

Z = lateral load per unit area.

Writing $\dfrac{T}{D} = \alpha^2$, $\dfrac{Z}{D} = p$; we have

$$\nabla_1^4 w + \alpha^2\nabla_1^2 w = p \tag{1.1}$$

For a clamped edge we shall have

$$w = 0,\ \frac{\partial w}{\partial r} = 0 \tag{1.2}$$

on the boundary.

If we enclose a portion of the plate bounded by a circle the normal shearing force N_r in the direction of w on an element are due to the action of the remaining part of the plate is given by

$$N_r = -\frac{\partial}{\partial r}\left[D\nabla_1^2 w + Tw\right] \tag{1.3}$$

2. Circular plate with clamped edge under concentrated load and uniorm compression

For a concentrated load equation (1.1) takes the form

$$(\nabla_1^2+\alpha^2)\nabla_1^2 w = 0. \tag{2.1}$$

If W be the load at the centre, considering a small circular area of radius r we have

$$\underset{r\to 0}{\text{Lt.}}\, r\frac{\partial}{\partial r}[\nabla_1^2+\alpha^2]w = \frac{W}{2\pi D} \tag{2.2}$$

using (1.3).

Let us assume

$$w = A+BJ_0(\alpha r)+C[\log r-Y_0(\alpha r)] \tag{2.3}$$

as a solution of equation (2.1).

Substituting w in equation (2.2)

$$C\alpha^2 = \frac{W}{2\pi D}$$

$$\therefore \qquad C = \frac{W}{2\pi T} \tag{2.4}$$

At the boundary $\qquad w = \dfrac{dw}{dr} = 0$ at $r = a$.

$$A+BJ_0(\alpha a)+C[\log a-Y_0(\alpha a)] = 0$$

$$B\alpha J'_0(\alpha a)+C[1/a-\alpha Y'_0(\alpha a)] = 0$$

$$\therefore \qquad B = \frac{-C[1/a-\alpha Y'_0(\alpha a)]}{\alpha J'_0(\alpha a)} \tag{2.5}$$

$$A = \frac{C[1/a-\alpha Y'_0(\alpha a)]}{\alpha J'_0(\alpha a)}.J_0(\alpha a)-C[\log a-Y_0(\alpha a)] \tag{2.6}$$

Using (2.4), (2.5), (2.6) and the relations

$$J'_0(x) = -J_1(x)$$

$$Y'_0(x) = -Y_1(x)$$

$$J_n(x)Y'_n(x)-J'_n(x)Y_n(x) = 1/x$$

We have

$$w = \frac{W}{2\pi T\alpha J_1(\alpha a)}[1/a\{1+J_0(\alpha r)-J_0(\alpha a)\}+\alpha Y_1(\alpha a)J_0(\alpha r)$$

$$-\alpha J_1(\alpha a)\{Y_0(\alpha r)-\log(\alpha r)+\log(\alpha a)\}] \tag{2.7}$$

Central defletcion :—

$$\frac{W}{2\pi T\alpha J_1(\alpha a)}[1/a\{2-J_0(\alpha a)\}+\alpha Y_1(\alpha a)-\alpha J_1(\alpha a)\log(\alpha a)] \tag{2-8}$$

$$= \frac{Wa^2}{16\pi D} \text{ as } T\to 0.$$

3. *Circular plate with clamped edge under symmetrical load and uniform compression.* In this problem equation (1.1)

takes the form

$$(\nabla_1^2+\alpha^2)\nabla_1^2 w = \frac{\varphi(r)}{D} = f(r)\,;\ \text{(say)} \tag{3.1}$$

Boundary conditions are as before

$$w = \frac{dw}{dr} = 0 \text{ at } r = a \tag{3.2}$$

Let us assume

$$w = \sum_{s=1}^{\infty} A_s[J_0(k_s r)-J_0(k_s a)] \tag{3.3}$$

k_s being the s^{th} root of

$$J_1(ka) = 0, \tag{3.4}$$

as the solution of (3.1).

It is evident that the boundary conditions are satisfied by the above equations.

Now substituting (3.3) in equation (3.1) we have

$$-\Sigma A_s k_s^2(\alpha^2-k_s^2)J_0(k_s r) = f(r) \tag{3.5}$$

If it is possible to expand $f(r)$ in a series of Bessel functions we get by virtue of (3.4)

$$\int_0^a \sum_{s=1}^{\infty} A_s k_s^2(k_s^2-\alpha^2)J_0^2(k_s r)r\,dr = \int_0^a f(r)J_0(k_s r)r\,dr$$

or,

$$A_s a^2 k_s^2(k_s^2-\alpha^2)\frac{J_0^2(K_s a)}{2} = \int_0^a f(r)J_0(K_s r)r\,dr$$

$\therefore$

$$A_s = \frac{2}{a^2 k_s^2(k_s^2-\alpha^2)J_0^2(k_s a)}\int_0^a f(r)J_0(k_s r)r\,dr \tag{3.6}$$

As an example let us suppose that the load varies as $(b^2-r^2)^{\frac{1}{2}}$ over a concentric circular area of radius $b < a$. [cf. Sen, 1935]

In this case we have

$$\left.\begin{aligned} f(r) &= C(b^2-r^2)^{\frac{1}{2}} \quad \text{when} \quad r < b < a \\ f(r) &= \quad 0 \quad\quad\quad\ \text{when} \quad b < r < a \end{aligned}\right\} \tag{3.7}$$

where C is a constant.

Therefore equation (3.6) becomes

$$A_s = \frac{2C}{a^2 k_s^2 (k_s^2 - \alpha^2) J_0^2(k_s a)} \int_0^a J_0(k_s r)(b^2 - r^2)^{\frac{1}{2}} r\,dr \tag{3.8}$$

putting $r = b \sin\theta$ we have

$$\int_0^a J_0(k_s r)(b^2 - r^2)^{\frac{1}{2}} r\,dr = b^3 \int_0^{\pi/2} J_0(k_s b \sin\theta) \cos^2\theta \sin\theta\,d\theta \tag{3.9}$$

Using the expansion for $J_0(k_s b \sin\theta)$ and integrating term by term the integral (3.9) becomes

$$b^3 F(k_s b) \tag{3.10}$$

where $$F(k_s b) = \tfrac{1}{3}\left[1 - \frac{k_s^2 b^2}{2\,5} + \frac{k_s^4 b^4}{2\,4.5.7} - \cdots\right] \tag{3.11}$$

Hence we have $$A_s = \frac{2b^3 C F(k_s b)}{a^2 k_s^2 (k_s^2 - \alpha^2) J_0^2(k_s a)} \tag{3.12}$$

$$\therefore \quad w = \sum_{s=1}^{\infty} A_s [J_0(k_s r) - J_0(k_s a)] \tag{3.13}$$

$$= \frac{2b^3 C}{a^3} \sum_{s=1}^{\infty} \frac{F(k_s b)}{k_s^2 (k_s^2 - \alpha^2) J_0^2(k_s a)} [J_0(k_s r) - J_0(k_s a)] \tag{3.14}$$

which is convergent.

Central deflection:—

$$\frac{2b^3 C}{a^3} \sum_{s=1}^{\infty} \frac{F(k_s b)}{k_s^2 (k_s^2 - \alpha^2)} [1 - J_0(k_s a)] \tag{3.14$'$}$$

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ON SELF-ASSOCIATE AND SELF-CONJUGATE AFFINE CONNECTIONS

By

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1. Consider an n-dimensional Riemannian space with the fundamental tensor g_{ij} which admits an arbitrary parallel displacement of vectors

ξ^t given by
$$d\xi^t + \overset{t}{\Gamma}_{ij}\xi^i dx^j = 0 \tag{1.1}$$

Let the covariant derivative with respect to (1.1) be denoted by a comma. Then the following facts are known (Sen, 1950).

If we put
$$a = \overset{t}{\Gamma}_{ij}, \quad a^* = \overset{t}{\Gamma}_{ij} + g^{mt}g_{im,j}, \quad a' = \overset{t}{\Gamma}_{ji}, \tag{1.2}$$
then for every parallelism corresponding to a, there exist uniquely two parallelisms corresponding to a^* and a', which are respectively called the *associate* and *conjugate* of a, having the involutory property

$$a^{**} = a'' = a.$$

Specially, a is self-associate if $a = a^*$ and self conjugate if $a = a'$. Further if we denote $\overset{t}{\Gamma}_{ij} - \overset{t}{\Gamma}_{ji} = \overset{t}{\Lambda}_{ij}$ and put

$$\left.\begin{aligned} \alpha &= g^{mt}g_{im,j}, \quad \alpha_c = g^{mt}g_{jm,i}, \quad \gamma = g^{mt}g_{ij,m}, \\ \beta &= g^{mt}g_{is}\overset{s}{\Lambda}_{mj}, \quad \beta_c = g^{mt}g_{js}\overset{s}{\Lambda}_{mi}, \quad \delta = g^{mt}g_{ms}\overset{s}{\Lambda}_{ij} = \overset{t}{\Lambda}_{ij} \end{aligned}\right\} \tag{1.3}$$

and construct the sequence

$$a_1 = a, \quad a_2 = a^*, \quad a_3 = a^{*\prime}, \quad a_4 = a^{*\prime *}, \ldots\ldots$$

then the sequence is a finite cyclic sequence of 12 terms having the following values

$$a_1 = a, \quad a_2 = a + \alpha, \quad a_3 = a + \alpha_c - \delta,$$
$$a_4 = a + \alpha - \gamma + \beta, \quad a_5 = a + \alpha_c - \gamma + \beta_c - \delta,$$
$$a_6 = a + \alpha + \alpha_c - \gamma + \beta + \beta_c, \quad a_7 = a + \alpha + \alpha_c - \gamma + \beta + \beta_c - \delta,$$
$$a_8 = a + \alpha_c - \gamma + \beta + \beta_c - \delta, \quad a_9 = a + \alpha - \gamma + \beta + \beta_c,$$
$$a_{10} = a + \alpha_c + \beta_c - \delta, \quad a_{11} = a + \alpha + \beta, \quad a_{12} = a - \delta.$$

Also the Christoffel symbol $\left\{ {t \atop ij} \right\} = \frac{1}{2}(a_i + a_{i+6}), \quad i = 1, 2, \ldots\ldots$

Or, writing u for $\left\{ {t \atop ij} \right\}$, it follows from the above values of the a's that

$$u = a + \tfrac{1}{2}(\alpha + \alpha_c - \gamma + \beta + \beta_c - \delta) \tag{1.4}$$

In the present paper the above results have been used to construct a sequence of self-associate affine connections and a sequence of self-conjugate affine connections, and to express the Christoffel symbol in terms of each of the sequences. Also certain other analogous properties of both the sequences have been obtained.

2. Let a be self-associate. Then $\alpha = \alpha_0 = \gamma = 0$. Writing b_1 for the self-associate a we obtain from (1 4) the equation

$$b_1 = u - \tfrac{1}{2}(\beta + \beta_c - \delta). \tag{2.1}$$

Obviously, the conjugate b_1' of b_1, as defined by (1.2), is

$$b_1' = u - \tfrac{1}{2}(\beta + \beta_c + \delta).$$

And, by definition (1.2), the associate $b_1'^*$ of b_1' is given by

$$b_1'^* = b_1' + g^{mt} g_{im;j}$$

where the semi-colon denotes covariant derivatives with respect to b_1'. By straight forward calculation it is seen that

$$b_1'^* = u + \tfrac{1}{2}(\beta - \beta_c + \delta).$$

Now define $$b_2 = \tfrac{1}{2}(b_1' + b_1'^*).$$

It is then easily seen that

$$b_2 = u - \tfrac{1}{2}\beta_c.$$

Similarly, define b_3 as $$b_3 = \tfrac{1}{2}(b_2' + b_2'^*).$$

It can then be seen that b_3 is given by

$$b_3 = u + \tfrac{1}{4}(\delta - \beta).$$

In the same manner

$$b_4 = u - \tfrac{1}{4}\beta_c - \tfrac{1}{8}(\delta - \beta), \quad b_5 = u + \tfrac{1}{8}\beta_c + \tfrac{3}{16}(\delta - \beta),$$

and so on. Evidently all the b's are self-associate. It may now be verified that if for $n = 1,2,3,4$ we write

$$b_n = u + T_n\beta_c + S_n(\delta - \beta),$$

then $T_n = (-1)^n \dfrac{t_n}{2^{n-2}}$,

where
$$\left.\begin{aligned} t_n &= 2t_{n-1} - 1, \quad \text{if } n \text{ is odd},\\ t_n &= 2t_{n-1} + 1, \quad \text{if } n \text{ is even},\\ \text{and} \quad t_0 &= \tfrac{3}{8} \end{aligned}\right\} \tag{2 2}$$

and
$$S_n = -T_{n+1}.$$

Thus
$$b_n = u + T_n\beta_c - T_{n+1}(\delta - \beta). \tag{2.3}$$

It follows from (2.2) that

$$T_n - T_{n+1} = (-1)^{n-1}\left(\frac{t_n}{2^{n-2}} + \frac{t_{n+1}}{2^{n-1}}\right)$$

and

$$T_n + T_{n+1} = (-1)^{n-1}\left(\frac{t_n}{2^{n-2}} - \frac{t_{n+1}}{2^{n-1}}\right)$$

Substituting the values of t's obtained from (2.2) (*i.e.* $t_1 = -\frac{1}{2}$, $t_2 = \frac{1}{2}$, $t_3 = 0$, $t_4 = 1$, $t_5 = 1$), it is seen that

$$T_n - T_{n+1} = 2\, T_{n+2} \tag{2.4}$$

$$T_n + T_{n+1} = -\frac{1}{2^{n-1}} \tag{2.5}$$

All these hold for the particular cases $n = 1,2,3$. We now deduce the following properties.

(1) Let us show that the formula (2.3) (together with the defining relation (2.2) and therefore with the relations (2.4), (2.5)) hold for all n. Assume the formula (2.3) to hold for any particular n. Then

$$b_n' = u + T_n\beta + T_{n+1}(\delta + \beta_o)$$

If a solidus denotes covariant derivative with respect to b'_n, it can be seen by straightforward calculation that

$$g_{ij/k} = -(T_n + T_{n+1})\,(g_{is}\overset{s}{\Lambda}_{jk} + g_{js}\overset{s}{\Lambda}_{ik}) \tag{2.6}$$

So

$$\begin{aligned} b_n'^{*} &= b_n' + g^{mt}g_{tm/j} \\ &= u + T_n\beta + T_{n+1}(\delta + \beta_o) - (T_n + T_{n+1})\,(\delta + \beta) \\ &= u - T_n\delta + T_{n+1}(\beta_o - \beta). \end{aligned}$$

Therefore

$$\begin{aligned} b_{n+1} &= \tfrac{1}{2}(b_n' + b_n'^{*}). \\ &= u + T_{n+1}\beta_o - \tfrac{1}{2}(T_n - T_{n-1})\,(\delta - \beta). \end{aligned}$$

Applying (2.4) we finally get

$$b_{n+1} = u + T_{n+1}\beta_o - T_{n+2}(\delta - \beta) \tag{2.7}$$

Hence, by mathematical induction, (2.3) holds for all n.

(2) Let us express Christoffel symbol, which is also self-associate, in relation to the terms of the sequence $\{b_n\}$. Subtracting (2.3) from (2.7) we have

$$b_{n+1} - b_n = (T_{n+1} - T_n)\beta_o + (T_{n+1} - T_{n+2})(\delta - \beta)$$

It therefore follows from (2.4) that

$$b_{n+1} - b_n = -2\{T_{n+2}\beta_o - T_{n+3}(\delta - \beta)\}.$$

But from (2.3) we have

$$b_{n+2} = u + T_{n+2}\beta_c - T_{n+3}(\delta - \beta).$$

Therefore
$$u = b_{n+2} + \tfrac{1}{2}(b_{n+1} - b_n). \tag{2.8}$$

This expresses Christoffel symbol in terms of 3 consecutive terms of the sequence $\{b_n\}$ of self associate affine connections.

(3) Torsion B_n of b_n is defined as $\frac{1}{2}(b_n - b_n')$. So

$$B_n = -T_{n+1}\delta + \tfrac{1}{2}(T_n - T_{n+1})(\beta_c - \beta).$$

Therefore by (2 4)

$$B_n = -T_{n+1}\delta + T_{n+2}(\beta_c - \beta) \tag{2.9}$$

where the T's have the values given by (2.2).

(4) As before, let the covariant derivative with respect to b_n and b_n' be denoted by a comma and a solidus respectively

Corresponding to the quantities α, α_c, γ given in (1.3), let us construct the quantities

$$\alpha_n' = g^{mt} g_{im|j}, \quad \alpha_n'{}_{/c} = g^{mt} g_{jm|i}, \quad \gamma_n' = g^{mt} g_{ij|m}.$$

It is then seen from (2.5) and (2.6) that

$$\alpha_n' = \frac{1}{2^{n-1}}(\beta + \delta), \; \alpha_n'{}_{/c} = \frac{1}{2^{n-1}}(\beta_c - \delta), \; \gamma_n' = -\frac{1}{2^{n-1}}(\beta + \beta_c) \tag{2.10}$$

3. In the last article we have obtained a sequence of self-associate affine connections and deduced some of its properties. In this article we shall obtain the corresponding results with regard to a sequence of self-conjugate affine connections. So let us suppose that a is self-conjugate. Then $\beta = \beta_c = \delta = 0$. Writing c_1 for the self-conjugate a we obtain from (1.4) the equation

$$c_1 = u - \tfrac{1}{2}(\alpha + \alpha_c - \gamma) \tag{3.1}$$

The associate c_1^* of c_1, as defined by (1.2), is given by

$$c_1^* = c_1 + \alpha = u + \tfrac{1}{2}(\alpha - \alpha_c + \gamma)$$

Define c_2 as
$$c_2 = \tfrac{1}{2}(c_1^* + c_1^{*\prime}).$$

Then c_2 is given by
$$c_2 = u + \tfrac{1}{2}\gamma.$$

Similarly define c_3 as
$$c_3 = \tfrac{1}{2}(c_2^* + c_2^{*\prime}).$$

Then it can be seen by straightforward calculation that

$$c_3 = u - \tfrac{1}{4}(\alpha + \alpha_c)$$

In the same manner

$$c_4 = u + \tfrac{1}{4}\gamma + \tfrac{1}{8}(\alpha + \alpha_c), \; c_5 = u - \tfrac{1}{8}\gamma - \tfrac{3}{16}(\alpha + \alpha_c),$$

and so on. Evidently all the c's are self-conjugate. Now comparing the values of $c_1, \ldots, c_5$ respectively with $b_1, \ldots b_5$ of §2, we see that for $n = 1, 2, 3, 4$, we can write

$$c_n = u - T_n\gamma + T_{n+1}(\alpha + \alpha_c) \tag{3.2}$$

where the T's have the values given by (2.2). We can then deduce the following properties as in §2.

(1) To see that the formula (3.2) holds for all n, let us assume the formula holds for any particular n. Denote the covariant derivative with respect to c_n by an ordinary bracket. Then by straightforward calculation and application of (2.4) we get

$$(g_{ij})_k = 2\{-T_{n+1}g_{ij,k}+T_{n+2}(g_{ik,j}+g_{jk,i})\}. \tag{3.3}$$

So $$c_n{}^* = c_n+g^{mt}(g_{im})_j = u+T_n\alpha_c-T_{n+1}(\alpha+\gamma). \tag{3.4}$$

Then $$c_{n+1} = \tfrac{1}{2}(c_n{}^*+c_n{}^{*\prime}) = u-T_{n+1}\gamma+T_{n+2}(\alpha+\alpha_c). \tag{3.5}$$

Hence, by mathematical induction, formula (3.2) holds for all n.

(2) To express Christoffel symbol, which is also self-conjugate, in relation to the terms of the sequence $\{c_n\}$, subtract (3.2) from) (3.5) and apply (2.1). We get

$$c_{n+1}-c_n = -2\{-T_{n+2}\gamma+T_{n+3}(\alpha+\alpha_c)\}.$$

But from (3.2), $$c_{n+2} = u-T_{n+2}\gamma+T_{n+3}(\alpha+\alpha_c).$$

Therefore $$u = c_{n+2}+\tfrac{1}{2}(c_{n+1}-c_n). \tag{3.6}$$

This expresses Christoffel symbol u in terms of 3 consective terms of the sequence $\{c_n\}$ of self conjugate affine connections. The result (3.6) for the particular case $n = 1$ was given by Sen (Sen, 1949).

(3) Covariant derivative of g_{ij} with respect to c_n is given by (3.3). If we construct the quantites

$$\alpha_n = g^{mt}(g_{im})_j,\ \alpha_{n/c} = g^{mt}(g_{jm})_i,\ \gamma_n = g^{mt}(g_{ij})_m,$$

it is seen that they are related to α, α_c, γ by relations

$$\left.\begin{aligned} \alpha_n &= 2\{-T_{n+1}\alpha+T_{n+2}(\alpha_c+\gamma)\}\\ \alpha_{n/c} &= 2\{-T_{n+1}\alpha_c+T_{n+2}(\alpha_c+\gamma)\}\\ \gamma_n &= 2\{-T_{n+1}\gamma+T_{n+2}(\alpha+\alpha_c)\}, \end{aligned}\right\} \tag{3.7}$$

(3) Torsion $C_n{}^*$ of $c_n{}^*$ is $\tfrac{1}{2}(c_n{}^*-c_n{}^{*\prime})$. So from (3.4) and (2.5) we get

$$C_n{}^* = \frac{1}{2^n}(\alpha-\alpha_c). \tag{3.8}$$

Properties (1) to (4) of §2 may be compared with properties (1) to (4) of §3.

4. Finally we give two applications of formulae (2.8) and (3.6). (1) Taking the torsion of (2.8), it follows from (3) of §2 that

$$0 = B_{n+2}+\tfrac{1}{2}(B_{n+1}-B_n).$$

By the application of (2.9) we get

$$2T_{n+4}(\beta_c-\beta)+T_{n+3}(\beta_c-\beta-2\delta)+T_{n+2}(\beta-\beta_c-\delta)+T_{n+1}\delta = 0 \tag{4.1}$$

Again, taking covariant derivatives of g_{ij} with regard to the same coordinate, say x^m, we get from (3.6) that

$$\begin{aligned} 0 = {} & \text{Covariant derivative of } g_{ij} \text{ with respect to } c_{n+2} \\ & + \tfrac{1}{2} (\text{Covariant derivative of } g_{ij} \text{ with respect to } c_{n+1}) \\ & - \tfrac{1}{2} (\text{Covariant derivative of } g_{ij} \text{ with respect to } c_{n}). \end{aligned}$$

Multiplying by g^{mt} and summing for m, it follows from (3) of §3 that

$$\gamma_{n+2} + \tfrac{1}{2}\gamma_{n+1} - \tfrac{1}{2}\gamma_n = 0.$$

Applying the third of the equations (3.7), we get

$$2T_{n+4}(\alpha + \alpha_c) + T_{n+3}(\alpha + \alpha_c - 2\gamma) - T_{n+2}(\alpha + \alpha_c + \gamma) + T_{n+1}\gamma = 0. \tag{4.2}$$

There are two other equations of this type.

(2) Let $C(a)$ denote the curvature tensor formed with coefficients of an affine connection a It is known (Sen, 1950) that if a, b, c, d are four sets of coefficients of affine connections such that

$$a \sim b = k(c \sim d)$$

where k is a scalar, then

$$C(\tfrac{1}{2}(a+b)) - \tfrac{1}{2}\{C(a) + C(b)\} = k^2[C(\tfrac{1}{2}(c+d)) - \tfrac{1}{2}\{C(c) + C(d)\}]$$

So, if R^t_{ijk} denotes the curvature tensor formed with Christoffel symbols, then it follows respectively from (2 8) and (3.6) that

$$4[C(\tfrac{1}{2}(u + b_{n+2})) - \tfrac{1}{2}\{R^t_{ijk} + C(b_{n+2})\}] = C(\tfrac{1}{2}(b_{n+1} + b_n)) - \tfrac{1}{2}\{C(b_{n+1}) + C(b_n)\} \tag{4.3}$$

$$4[C(\tfrac{1}{2}(u + c_{n+2})) - \tfrac{1}{2}\{R^t_{ijk} + C(c_{n+2})] = C(\tfrac{1}{2}(c_{n+1} + c_n)) - \tfrac{1}{2}\{C(c_{n+1}) + C(c_n)\}. \tag{4.4}$$

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NEWMANIAN GEOMETRY—I

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1. Introduction. Metric spaces are well-known to day. A set S in which a distance $d(a, b)$ between a pair of elements a, b of S is defined such that it satisfies the postulates (*i*) $d(a, b)$ is a real number ≥ 0, (*ii*) $d(a, b) = d(b,a)$, (*iii*) $d(a, b)=0$ if and only if $a = b$, (*iv*) triangle inequality: $d(a, b)+d(b, c) \geq d(a, c)$, is called a metric space. But now-a-days metric spaces have been considered in which the distances are not necessarily real or complex numbers. Indeed, it has been shown [Blumenthal (1953), Ellis (1951*a*)] that if with each pair of elements a, b of a Boolean algebra B, we associate the element $a*b = ab'+a'b$ and refer to it as the distance between a, b, then this association satisfies $(M_1): a*b = b*a$, $(M_2): a*b = 0$ if and only if $a = b$, (M_3): triangle inclusion $a*b + b*c \geq a*c$. The Boolean algebra is thus metrized or becomes a metric space. As the distance here is an element of B (which is not a real or a complex number), B is called an autometrized Boolean algebra or a Boolean geometry. We shall call it a Boolean space.

Further, this idea has been generalised and applied to the Brouwerian algebra [Nordhaus and Lapidus (1954).] Here the distance between a pair of elements a, b of a Brouwerian algebra is defined to be the element $a*b = (a-b)+(b-a)$ and this satisfies conditions M_1, M_2 and M_3. Thus a Brouwerian algebra becomes an autometrized Brouwerian space.

There exists now a days another important algebra discovered by Newman [Newman, (1942, 1944); Birkhoff (1948)] which is akin to a Boolean algebra. This algebra is known as a Newman algebra. The aim of this paper is to study autometrized Newman algebra which we call a Newmanian space. It is important to note that a Newman algebra is not a lattice; but it contains a lattice namely a Boolean algebra. It also contains a Boolean naring and is the direct sum of these two. Consequently we get a Boolean space and a line which will be called the principal axis of the Newmanian space. Thus a Newmanian space is the cartesian product of a Boolean space and a Newmanian line. One can consider many important properties of this space. Some interesting properties of the space like metric betweenness and linearity have been discussed in this paper.

2. General properties of a Newman algebra. Definition 1. *An algebra N with two binary operations satisfying the following postulates:*

1. $a(b+c) = ab+ac$, $(a+b)c = ac+bc$,
2. *There exists* 1 *such that* $a1 = a$, *for all* $a \in N$,
3. *There exists* 0 *such that* $a+0 = 0+a = a$, *for all* $a \in N$,

4. *For each a, there exists a' such that $a+a' = 1$ and $aa' = 0$, is called a Newman algebra.*

For all a, $a^2 = a$, $a'' = a$, $a0 = 0$ and a' is unique. If $1 = 0$, the system is trivial. For non-trivial systems $1 \neq 0$. All elements of N are not additively idempotent. Hence one can take $1+1 = 2 \neq 1$. $2+2 = 2$. We call left multiples $y2$ of 2, as even elements. Even elements are additively idempotent and conversely. Left or right multiples of even elements are even. If x is even, then $x2' = 0$ and $x+1 = 1+x = 1$. The even elements of N will form a Boolean algebra B with 0 and 2 as the least and greatest elements. If $2'$ is the complement of 2 in N, then left or right multiples $y2'$ or $2'y$ of 2, are called odd elements. For an odd element y, $y+y = 0$ and $y2' = y$. The odd elements of N will form a set E which is a Boolean naring of characteristic 2. Also 0 is the only element which is both even and odd.

N is the direct sum of the Boolean algebra B of even elements and the Boolean naring E of odd elements; *i e.*, for any element $n \in N$, $n = b+e$ where $b \in B$ and $e \in E$. Also if $n_1 = b_1+e_1$, $n_2 = b_2+e_2$, then $n_1+n_2 = (b_1+b_2)+(e_1+e_2)$ and $n_1n_2 = (b_1b_2)+(e_1e_2)$. Addition and multiplication are commutative in N as they are so in both B and E. Addition is associative in N as B and E are additively associative. A Newman algebra in which every element is additively idempotent is a Boolean algebra.

Theorem 2.1. *Every element of E has a unique complement with respect to 0 and $2'$ in E,*

Proof. E is an additive abelian group. The solution of $e+x = 2'$ exists for any $e \in E$, $x = e+2'$ being the solution. Again $e(e+2') = e+e2' = e+e = 0$; so $e+2'$ is the complement of e in E with respect to 0 and $2'$. This is unique as the solution is unique.

Theorem 2.2. *The complement of any element $n = b+e$ of N, $b \in B$, $e \in E$, is $n' = \bar{b}+e^\times$ where $\bar{b}$ is the complement of b in B with respect to 0 and 2 and $e^\times$ is that of e in E with respect to 0 and $2'$.*

Proof. $(b+e)+(\bar{b}+e^\times) = (b+\bar{b})+(e+e^\times) = 2+2' = 1$

and $(b+e)(\bar{b}+e^\times) = (b\bar{b})+(ee^\times) = 0.$

Theorem. 2.3. *The subset $b+E$ of N where $b \in B$, is a Newman algebra of characteristic 2.*

Proof. The sum and the product of any two elements $b+e_1$, $b+e_2$ of $b+E$ belong to $b+E$. Distributive laws hold as they are elements of N. Also $(b+e_1)+(b+0) = b+e_1$ and $(b+e_1)(b+2') = b+e_1$; so $b+0$ and $b+2'$ act as 0 and 1. Again $(b+e_1)(b+e_1^\times) = b+0$ and $(b+e_1)+(b+e_1^\times) = b+2'$. Also $(b+e_1)+(b+e_1) = b+0$, which shows that the elements are of charcteristic 2.

Cor. 2.31. *The Boolean naring E is a Newman algebra (of odd elements) of characteristic 2 and conversely.*

It is evident that the representation of any element n of N as $n = b+e$, $b \in B$, and $e \in E$ is unique; *i.e.*, if $n = b_1+e_1 = b_2+e_2$, then $b_1 = b_2$, $e_1 = e_2$.

Theorem 2.4. If $n_1 = b_1+e_1$, $n_2 = b_2+e_2$; $b_1, b_2 \in B$; $e_1, e_2 \in E$;

then $$n_1n'_2+n'_1n_2 = (b_1\bar{b}_2+\bar{b}_1b_2)+(e_1\overset{\times}{e}_2+\overset{\times}{e}_1e_2)$$

$$= (b_1\bar{b}_2+\bar{b}_1b_2)+(e_1+e_2)$$ where n' is the complement of n in N with respect to 0 and 1.

Proof. $n_1n'_2+n'_1n_2 = (b_1+e_1)(b_2+e_2)'+(b_1+e_1)'(b_2+e_2)$

$$= (b_1+e_1)(\bar{b}_2+\overset{\times}{e}_2)+(\bar{b}_1+\overset{\times}{e}_1)(b_2+e_2)$$

$$= (b_1\bar{b}_2+e_1\overset{\times}{e}_2)+(\bar{b}_1b_2+\overset{\times}{e}_1e_2) = (b_1\bar{b}_2+\bar{b}_1b_2)+(e_1\overset{\times}{e}_2+\overset{\times}{e}_1e_2)$$

$$= (b_1\bar{b}_2+\bar{b}_1b_2)+\{e_1(e_2+2')+(e_1+2')e_2\}$$

$$= (b_1\bar{b}_2+\bar{b}_1b_2)+(e_1+e_2).$$

Definition 2. *We define an inclusion relation in N by the condition $a \geq b$ if and only if $ab = b$.*

Theorem 2.5. *If $n_1 = b_1+e_1$, $n_2 = b_2+e_2$; $b_1, b_2 \in B$; $e_1, e_2 \in E$; then $n_1 \geq n_2$ if and only if $b_1 \geq b_2$, $e_1 \geq e_2$.*

Proof. $n_1n_2 = (b_1+e_1)(b_2+e_2) = b_1b_2+e_1e_2$. So, from the definition, $n_1 \geq n_2$ if and only if $n_1n_2 = n_2$; *i.e.*, if and only if $b_1b_2+e_1e_2 = b_2+e_2$; *i.e.*, if and only if $b_1b_2 = b_2$, $e_1e_2 = e_2$; *i.e.*, if and only if $b_1 \geq b_2$, $e_1 \geq e_2$.

Cor. 2.51. If $b_1 \geq b_2$; $b_1, b_2 \in B$, then for any $e \in E$, $b_1+e \geq b_2+e$.

Proof. As $e^2 = e$, so, $e \geq e$ and the corollary follows at once from above theorem.

Definition 3. *We define an operation $*$ in N given by $n_1*n_2 = n_1n'_2+n'_1n_2$ $= b_1\circ b_2+(e_1+e_2)$, writing $b_1\circ b_2 = b_1\bar{b}_2+\bar{b}_1b_2$. The operation $\circ$ is a metric operation in the Boolean algebra B* [Blumenthal (1953); Ellis (1951*b*)].

Theorem 2.6. *The operation $*$ is a metric operation in N.*

Proof. $n_1 * n_2 = n_1n'_2+n'_1n_2$ and $n_2 * n_1 = n_2n'_1+n'_2n_1$.

As addition and multiplication are commutative in N, $n_1*n_2 = n_2*n_1$.

Also if $n_1=n_2$, then $n_1*n_2 = n_1n'_1+n'_1n_1 = 0$.

Again, if $n_1*n_2 = 0$, then $n_1n'_2+n'_1n_2 = b_1\circ b_2+(e_1+e_2) = 0 = 0+0$.

So $b_1\bullet b_2 = 0 \ldots (1)$ and $e_1+e_2 = 0 \ldots (2)$. From (1) it follows that $b_1 = b_2$ (as B is autometrized by the operation $\bullet$). From (2), $e_1 = e_2$ (as E is a naring of characteristic 2).

So $n_1*n_2 = 0$ if and only if $n_1 = n_2$.

If again $n_3 = b_3+e_3$, then $n_1*n_2+n_2*n_3 = \{(b_1\circ b_2)+(e_1+e_2)\}+\{(b_2\circ b_3)+(e_2+e_3)\}$
$= (b_1\circ b_2+b_2\bullet b_3)+(e_1+e_3)$
$\geq b_1\circ b_3+(e_1+e_3)$ (as $b_1\bullet b_2+b_2\bullet b_3 \geq b_1\circ b_3$ in B and if $b_1 \geq b_2$ then $b_1+e \geq b_2+e$).
$= n_1*n_3$.

So, $n_1{*}n_2 + n_2{*}n_3 \geq n_1{*}n_3$, which proves the triangle inclusion.

3. The elements of N can be considered as points; $n_1{*}n_2$ is proved above to satisfy the axioms of a distance. We write $n_1{*}n_2 = d(n_1, n_2)$; consequently the Newman algebra N is a metric space. As $d(n_1, n_2)$ is an element of N, N is autometrized.

Definition 4. *A Newmanian space is a metric space whose points are elements of the Newman algebra and whose distance function* is $d(n_1, n_2) = n_1{*}n_2 = n_1n'_2 + n'_1n_2$. *The geometry over this space is called Newmanian geometry.*

Remark. As $d(n_1, n_2) = n_1{*}n_2 = b_1 \circ b_2 + (e_1 + e_2)$, Putting $e_1 = 0 = e_2$, we get

$$d(b_1, b_2) = b_1{*}b_2 = b_1 \circ b_2 \in B.$$

Similarly, $d(e_1, e_2) = e_1{*}e_2 = e_2 + e_2 \in E$.

So B and E are autometrized by $*$ and it reduces to $\circ$ and $+$ which are metric operations in B and E respectively.

Also, $d(n_1, n_2) = (b_1{*}b_2) + (e_1{*}e_2) = d(b_1, b_2) + d(e_1, e_2)$.

Definition 5. *If* $n_1, n_2\, n_3$ *are three points such that* $d(n_1, n_2) + d(n_2, n_3) = d(n_1, n_3)$, *then* n_1, n_2, n_3, *are said to be linear. They are said to lie on a line.* Thus any three points of a line are linear.

Theorem 3.1. $b + E$ of N, $b \in B$ will form a line of N.

Proof. If $b + e_1, b + e_2, b + e_3$ are any three elements of $b + E$, then $(b + e_1){*}(b + e_2) + (b + e_2){*}(b + e_3) = \{(b{*}b) + (e_1{*}e_2)\} + \{(b{*}b) + (e_2{*}e_3)\} = (e_1 + e_2) + (e_2 + e_3) = e_1 + e_3 = (b + e_1){*}(b + e_3)$.

So any three points of $b + E$ are linear. This proves the theorem.

Cor 3.11. *E forms a line of N.*

Remark. As a Boolean naring with 1 is a Newman algebra of characteristic 2 like E, we get

Theorem 3.2. *A Boolean naring with 1 is a Newmanian space which consists of a line only where the distance is defined by* $d(a, b) = a{*}b = a + b$.

Definition 6. *In* N, *a subset* S_1 *is defined to be orthogonal to another subset* S_2 *if and only if* $n_1n_2 = 0$ *for every* $n_1 \in S_1$ *and* $n_2 \in S_2$. $eb = 0$ *for every* $e \in E$ *and* $b \in B$. So E is orthogonal to B. $b + E(b \neq 0)$ is not orthogonal to B; for, $(b + e)b = b + 0 = b \neq 0$. Both E and B contain 0

As, for any element n, $n = b + e$; $b \in B$, $e \in E$, it follows that any point n in the space N has two co-ordinates one $b \in B$ and the other $e \in E$. We may write this point as (b, e). The representation $n = b + e$ is similar to the representation of a complex number in the form $x + iy$. If $b = 0$, then $n \in E$ and if $e = 0$, then $n \in B$,

Definition 7. *E is called the principal Newmanian axis and $b + E$ is called a Newmanian-axis. The autometrized Boolean algebra B is called the Boolean spaae.*

So, from above, we get

Theorem 3.3. *A Newmanian space N is the Cartesian product of a Boolean space B and a principal axis E.*

In a Boolean algebra B, the operation $*$ is a group operation. In fact Elliot [Elliot (1953)] has observed that it is the only metric-cum-group operation in a Boolean algebra. Also in E, since $e_1*e_2 = e_1+e_2$, $*$ is a group operation. Now in N, $n_1*n_2 = b_1*b_2+e_1*e_2$. So $*$ is a group operation in N. Thus

Propositon. 3.1. *The operation $*$ is both a metric and a group operation in N.*

0 is the unit element of this group operation; for, $n*0 = n1+n'0 = n+0 = n$. This shows that $d(0, n) = n$. The element 0, therefore, acts as origin.

4. We shall regard a triple of elements n_1, n_2, n_3 as the vertices of a triangle with sides n_1*n_2, n_2*n_3 and n_3*n_1. It follows from theorem 2.6 that the sides of a triangle satisfy triangle inclusion. The symbol $\Delta(n_1, n_2, n_3)$ will be used to denote a triangle whose vertices are n_1, n_2, n_3.

Definition 8. *If x, y, z are the sides of a triangle, then the triangle $\Delta(x, y, z)$ will be called the first distance triangle of the original triangle. Similarly, the triangle $\Delta(y*z, z*x, x*y)$ will be called the second distance triangle and so on.*

Definition 9. *If each vertex of a triangle is equal to the opposite side, then the triangle is said to have fixity.*

Theorem 4.1. *If one vertex of a triangle is equal to the opposite side, then the triangle has fixity.*

Proof. Suppose for $\Delta(n_1, n_2, n_3), n_1*n_2 = n_3$; since $*$ is a metric and group operation, $n_2 = n_1*n_3$ and $n_2*n_3 = (n_1*n_3)*n_3 = n_1$

Cor. 4.11. *$\Delta(n_1, n_2, n_1*n_2)$ has fixity.*

Since $*$ is a group operation, given n_1 and n_3, there is exactly one element n_2 of N with $n_1*n_2 = n_3$ [*i.e.*, with $d(n_1, n_2) = n_3$]. So the space N contains one and only one point with an assigned distance from any chosen point. So N contains no isosceles triangle.

Theorem 4.2. *In N, any first distance triangle has fixity ond conversely.*

Proof. If $\Delta(n_1, n_2, n_3)$ is the original triangle, then $x = n_1*n_2, y = n_2*n_3, z = n_3*n_1$ are the sides. So $\Delta(x, y, z)$ is the first distance triangle

Now $x*y = (n_1*n_2)*(n_2*n_3) = n_1*n_3 = z$. Similarly, $y*z = x$ and $z*x = y$.

Conversely, if $\Delta(n_1, n_2, n_3)$ has fixity then $n_1*n_2 = n_3$ and $\Delta(n_1, n_2, n_3)$ is the first distance triangle of $\Delta(n_1, 0, n_2)$.

Definition 10. *A subspace of N is a subset such that the distance between any two elements of the subset belongs to it.* Evidently any two elements a and b together with their distance $a*b$ and 0 will form a subspace of N. So the subspace of a Newmanian space generated by two elements contains at most four points. Subspaces of N are subgroups with respect to the composition $*$ whose elements are all of order 2. Every subspace of N must contain 0.

Theorem 4.3 *B and E are subspaces of N.*

For, $b_1 * b_2 = b_1 \circ b_2 \in B$ and $e_1 * e_2 = e_1 + e_2 \in E$.

Remark. As $d(n_1, n_2) = d(b_1, b_2) + d(e_1, e_2)$, the distance $d(n_1, n_2)$ between any two points n_1 and n_2 has two components, one along the Boolean space B and the other along the principal axis. These components are distances between the corresponding projections on them.

Theorem 4.4. *If E_1 is a subspace of E and B_1 that of B, then $B_1 + E_1$ is a subspace of N.*

Proof. If $\gamma_1 = (\beta_1, \epsilon_1)$, $\gamma_2 = (\beta_2, \epsilon_2)$ are any two points of $B_1 + E_1$ where $\beta_1 \beta_2 \in B_1$ and $\epsilon_1, \epsilon_2 \in E_1$, then $d(\gamma_1, \gamma_2) = d(\beta_1, \beta_2) + d(\epsilon_1, \epsilon_2) \in B_1 + E_1$ (as B_1 and E_1 are subspaces).

5. Projection. *The coorrespondence $\varphi : n \rightarrow n2$ maps N on to B homomorphically.*

For, if $x, y \in N$, then $(x+y)2 = x2 + y2$ and $(x2)(y2) = (x+x)(y+y) = (xy+xy) + (xy+xy) = xy2 + xy2 = xy2$. Also, $(x2)2 = x2 + x2 = x2 = x + x = x$ if $x \in B$. So the homomorphism φ is an endomorphism of N on to B such that $\varphi^2 = \varphi$, *i.e.*, φ is idempotent and, therefore, is a projection of N on B. By this projection $(b+E)$ is mapped on to b in B.

Definition 11. *The image on B of any element $n \in N$* under φ *is called the projection of n on B.* The Newmanian axis $b+E$ which is mapped on to $b \in B$, is a projecting ray. So we get

Theorem 5.1. If one projects the space N on B, the projecting rays are Newmanian axes.

It can be observed that for any point of the axis $b_1 + E$, the Boolean co-ordinate is b_1; so the equation of the axis $b_1 + E$ can be written as $b = b_1$.

Similarly, the mapping $\psi : n \rightarrow n2'$ is an endomorphism of N on to E.

For, if $n_1 = b_1 + e_1, n_2 = b_2 + e_2$, then $(n_1 + n_2)2' = n_1 2' + n_2 2'$ and $(n_1 2')(n_2 2') = \{(b_1 + e_1)2'\}\{(b_2 + e_2)2'\} = e_1 e_2 = (b_1 b_2 + e_1 e_2)2' = (n_1 n_2)2'E$. Also $(e2')2' = e2' = e$. So $\psi^2 = \psi$, which shows that ψ is a projection of N on E. $B + e$ is projected on e by ψ. The image of n under ψ is called the projection of n on E.

It is to be noted that $B + e$ is not a line. We shall discuss about $B + e$ in another paper.

6. Metric betweenness. Definition 12. In N, n_2 is defined to be metrically between n_1 and n_3 if and only if $d(n_1, n_2) + d(n_2, n_3) = d(n_1, n_3)$. The relation is symbolically written as $(n_1, n_2, n_3)M$. It is a betweenness relation in the wide sense (not in the usual sense). It fails to have in many cases the special inner point property [Blumenthal (1953), p. 53].

Indeed, $(n_1, n_2, n_3)M$ and $(n_1, n_3, n_2)M$ may persist without having $n_2 = n_3$. It is clear that the relation has the other basic property namely symmetry of the outer points viz $(n_1, n_2, n_3)M$ if and only if $(n_3, n_2, n_1)M$. Nevertheless we adopt this terminology in this wide sense.

Theorem 6 1. *n_2 is metrically between n_1 and n_3 if and only if their corresponding projections on B are so.*

Proof. Suppose $n_1 = b_1+e_1, n_2 = b_2+e_2, n_3 = b_3+e_3$ where b_1, b_2, b_3 are the corresponding projections on B. Then, $(n_1, n_2, n_3)M$ if and only if $n_1{*}n_2+n_2{*}n_3 = n_1{*}n_3$; i.e if and only if $\{(b_1{*}b_2)+(e_1{*}e_2)\}+\{(b_2{*}b_3)+(e_2{*}e_3)\} = (b_1{*}b_3)+(e_1{*}e_3)$; i e., if and only if $\{(b_1{*}b_2)+(b_2{*}b_3)\}+(e_1{*}e_2+e_2{*}e_3) = (b_1{*}b_3)+(e_1{*}e_3)$; i e., if and only if $\{(b_1{*}b_2)+(b_2{*}b_3)\}+(e_1+e_3) = (b_1{*}b_3)+(e_1+e_3)$; i.e , if and only if $b_1{*}b_2+b_2{*}b_3 = b_1{*}b_3$; i e., if and only if $(b_1, b_2, b_3)M$.

Proposition 6.1. *If $(n_1, n_2, n_3)M$ holds, then three points n_1, n_2, n_3 are linear; and conversely, if three points are linear, one of them is metrically between the other two.*

[The proof follows from the definitions of linearity and metric betweenness].

Cor. 6.11. *If b_1, b_2, b_3 are linear in B, then any three points of the axes* b_1+E, b_2+E and b_3+E, each lying on an axis are linear.

Proof. From the above theorem, it follows immediately.

Any three points of two axes b_1+E and b_2+E are linear. For, if the three points lie upon one axis, then it is obvious. If two points say $n_1 = (b_1, e_1), n_2 = (b_1, e_2)$, lie upon the first axis and the other point $n_3 = (b_2, e_3)$ lies upon the second one, then $(b_1+e_1){*}(b_1+e_2)+(b_1+e_2){*}(b_2+e_3) = \{b_1{*}b_1+(e_1+e_2)\}+\{(b_1{*}b_2)+(e_2+e_3)\} = (0+b{*}b_2)+(e_1+e_3) = (b_1+e_1)\ (b_2+e_3)$; i.e., $d(n_1, n_2)+d(n_2, n_3) = d(n_1, n_3)$. Similarly is the proof for one point lying on the first axis and two points lying on the 2nd one. So we get

Theorem 6.2. *The join $(b_1+E)\cup(b_2+E)$ of two axis b_1+E and b_2+E forms a line.*

Theorem 6.3. *If b_1, b_2, b_3 are linear in B, then $(b_1+E)\cup(b_2+E)\cup(b_3+E)$ is a line of N.*

Proof. From above it follows that any three points on any two of the axes b_1+E, b_2+E, b_3+E are linear. Also from Cor. 6.11 it follows that if the three points lie on the three axes then also they are linear. This shows that any three points of $(b_1+E)\cup(b_2+E)\cup(b_3+E)$ are linear; so it forms a line.

It is to be noted that the join $(b_1+E)\cup(b_2+E)$ of the axes b_1+E and b_2+E is not a Newman algebra with respect to the compositions of N. But $E\cup(b+E)$ is a Newman algebra which is the dirct sum of $[0, b]$ and E. Hence it is a subalgebra of N as well as a subspace of the space N. $b+E$ is not a subspace although it is a Newman algebra; but $0+E = E$ is a subspace.

Theorem 6.4. *Three linear points of N fail to have special innerpoint property if and only if they lie upon one or two Newmanian axes.*

Proof. If $n_1 = b_1+e_1, n_2 = b_2+e_2, n_3 = b_3+e_3$; $(n_1. n_2. n_3)M$ and $(n_1, n_3\ n_2)M$ hold where $n_2 \neq n_3$, then we have $(b_1,b_2,b_3)M$ and $(b_1,b_3,b_2)M$ (from theorem 6.1); i e., $b_1{*}b_2+b_3{*}b_3 = b_1{*}b_3$...(1) and $b_1{*}b_3+b_3{*}b_2 = b_1{*}b_2$...(2). From (1) and (2), $b_1{*}b_2+b_2{*}b_3+b_3{*}b_2 = b_1{*}b_2$, Or, $b_1*b_2 + b_2*b_3 = b_1*b_2$ [as $b_2*b_3 = b_3*b_2$]; so from (1), $b_1*b_3 = b_1*b_2$. As * group operation, either $b_1 = b_2 = b_3$ or $b_3 = b_2$. So the three points lie on one or two axes.

The converse case can be proved easily by reversing the argument; for, if n_1, n_2, n_3 lie on one or two axes; *i.e.*, if $b_1 = b_2 = b_3$ or $b_2 = b_3 \neq b_1$, then (1) and (2) hold and so $(n_1, n_2, n_3)M$ and $(n_1, n_3, n_2)M$ hold.

Definition 13. *If a betweenness relation R has the property that $(p, q, r)R$ and $(p, x, q)R$ imply $(x, q, r)R$, the relation is said to have transitivity t_1 If the same two relations imply $(p, x, r)R$, the relation is said to have transitivity t_2.*

Remark. In an autometrized Boolean algebra B metric betweenness and lattice betweenness coincide [Blumenthal (1953); Ellis (1951)]. Evidently metric betweenness has t_1 and t_2 in B [Nordhaus and Lapidus, (1954), theorems 3.6 and 3.7]; *i.e.*, $(b_1, b_2, b_3)M$ and $(b_1, b, b_2)M$ imply $(b, b_2, b_3)M$ and $(b_1, b, b_3)M$.

Theorem 6.5. *Metric betweenness in N has the transitivity t_1 and transitivity t_2.*

Proof. Suppose $n_1 = b_1 + e_1$, $n_2 = b_2 + e_2$, $n_3 = b_3 + e_3$, $n = b + e$, and $(n_1, n_2, n_3)M$, $(n_1, n, n_2)M$ hold. Evidently we have $(b_1, b_2, b_3)M$ and $(b_1, b, b_2)M$ (from theorem 6.1). So, from the above remark, $(b, b_2, b_3)M$ and $(b_1, b, b_3)M$. Therefore $(n, n_2, n_3)M$ and $(n_1, n, n_3)M$ hold (theorem 6.1).

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LOVE WAVES IN GRANULAR MEDIUM

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1. **Introduction.** The mechanics of granular medium has been the subject of much investigation in recent times because of its application in soil mechanics, mining engineering, petroleum prospecting, flow of fluids through sand, packing of granular aggregrate and so on. The first investigation by Hara (1935) as extended and experimentally verified by Iida (1938, 1939) as well as subsequent works of Takahashi and Satô (1949, 1950) and Gassmann (1951, 1953) were all based on Hertz's theory of contact of elastic spheres. The experimental work of Hughes and his coworkers (Hughes and Jones, 1950; Hughes and Cross, 1951; Hughes and Kelly 1952) and Matsukawa and Hunter (1956) may be mentioned in this connection. A study by Bandt (1955), though essentially the same as those of previous investigators, strikes a new line and agrees more or less with the observations of Nasu (1940) and Fatt (1953, 1957). By generalizing Hertz's theory in which only the normal component of the contact force between the spheres are taken into account, Cattaneo (1938), Mindlin (1949), Mindlin and Deresiewicz (1953) and Johnson (1955) have tried to introduce varying oblique and tangential forces. An incremental stress-strain relation has also been proposed by Duffy and Mindlin (1957). Details of these investigations and some references on the subject may be found in the review article by Deresiewicz (1958). By a quite different approach with the statistical idea of the stochastic process Litwiniszyn (1956, 1957*a*, 1957*b*) has attempted to describe a granular medium and calls it a stochastic medium. Oshima (1954, 1955) on the other hand uses the concept of averages to reduce the granular medium to a continuous medium by introducing additional physical constants. In the present paper the theory of Oshima has been applied to discuss the propagation of Love waves in the granular medium. By a procedure similar to those in the theories of Love waves in elastic or hypoelastic bodies (Paria, 1958) the wave-velocity equation has been obtained in the form of a determinant of ninth order. The roots of this equation are in general complex and the imaginary part of an appropriate root measures the attenuation of the waves. The numerical solution of the wave-velocity equation will be given in a subsequent paper.

2. **Basic Equations.** The state of deformation in the granular medium is described by the displacement vector $\vec{u}(u, v, w)$ of the centre of gravity of a grain and the rotation vector $\vec{\xi}(\xi, \eta, \zeta)$ of the grain about its centre of gravity (Oshima, 1954, 1955). Let $\tau_{xx}, \tau_{xy}, \tau_{xz}$ denote the components of the resultant force acting on a surface element

of unit area perpendicular to the x-axis. The strese tensor τ_{ij} written in Cartesian coordinates as

$$\begin{pmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{pmatrix} \tag{1}$$

is not symmetrical, that is, $\tau_{ij} \neq \tau_{ji}$ However this can be written as the sum of a symmetrical and an antisymmetrcal tensor as

$$\tau_{ij} = \sigma_{ij} + \sigma'_{ij} \tag{2}$$

where

$$\sigma_{ij} = \tfrac{1}{2}(\tau_{ij} + \tau_{ji}) \tag{3}$$

$$\sigma'_{ij} = \tfrac{1}{2}(\tau_{ij} - \tau_{ji}). \tag{4}$$

The symmetric tensor $\sigma_{ij} = \sigma_{ji}$ is related to the symmetric strain tensor

$$e_{ij} \equiv e_{ji} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

by the Hooke's law and thus in isotropic medium we have

$$\begin{aligned} \sigma_{xx} &= \lambda e + 2\mu e_{xx} & \sigma_{yz} &= 2\mu e_{yz} \\ \sigma_{yy} &= \lambda e + 2\mu e_{yy} & \sigma_{zx} &= 2\mu e_{zx} \\ \sigma_{zz} &= \lambda e + 2\mu e_{zz} & \sigma_{xy} &= 2\mu e_{xy} \end{aligned} \tag{6}$$

where

$$e = e_{xx} + e_{yy} + e_{zz} \tag{7}$$

and (λ, μ) are Lamé constants. The anti-symmetric stresses σ'_{ij} are related to the rotation components (ξ, η, ζ) as

$$\sigma'_{yz} = -A\frac{\partial \xi}{\partial t}, \; \sigma'_{zx} = -A\frac{\partial \eta}{\partial t}, \; \sigma'_{xy} = -A\frac{\partial \zeta}{\partial t} \tag{8}$$

where A is the coefficient of friction between the individual grains and t denotes time. The relations

$$\sigma'_{xx} = \sigma'_{yy} = \sigma'_{zz} = 0 \tag{9}$$

hold by definition. Besides the non-symmetric stress tensor τ_{ij} there is also the non-symmetric stress couple or the secondary stress tensor M_{ij} with components

$$\begin{pmatrix} M_{xx} & M_{yx} & M_{zx} \\ M_{xy} & M_{yy} & M_{zy} \\ M_{xz} & M_{yz} & M_{zz} \end{pmatrix} \tag{10}$$

which is related to a non-symmetric strain tensor or the secondary strain tensor γ_{ij} as

$$M_{ij} = B\gamma_{ij} \tag{11}$$

where B is a third elastic constant. The tensor γ_{ij} is defined as

$$\gamma_{xx} = \frac{\partial}{\partial x}(\varpi_x+\xi),\ \gamma_{yy} = \frac{\partial}{\partial y}(\varpi_y+\eta),\ \gamma_{zz} = \frac{\partial}{\partial z}(\varpi_z+\zeta)$$

$$\gamma_{yz} = \frac{\partial}{\partial y}(\varpi_z+\zeta),\ \gamma_{zx} = \frac{\partial}{\partial z}(\varpi_x+\xi),\ \gamma_{xy} = \frac{\partial}{\partial x}(\varpi_y+\eta) \tag{12}$$

$$\gamma_{zy} = \frac{\partial}{\partial z}(\varpi_y+\eta),\ \gamma_{xz} = \frac{\partial}{\partial x}(\varpi_z+\zeta),\ \gamma_{yx} = \frac{\partial}{\partial y}(\varpi_x+\xi)$$

where
$$\varpi_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \varpi_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \varpi_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{13}$$

The six equations of motion are

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \varrho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \varrho\frac{\partial^2 v}{\partial t^2} \tag{14}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \varrho\frac{\partial^2 w}{\partial t^2}$$

and

$$\tau_{yz} - \tau_{zy} + \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_{zx}}{\partial z} = 0$$

$$\tau_{zx} - \tau_{xz} + \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{zy}}{\partial z} = 0 \tag{15}$$

$$\tau_{xy} - \tau_{yx} + \frac{\partial M_{xz}}{\partial x} + \frac{\partial M_{yz}}{\partial y} + \frac{\partial M_{zz}}{\partial z} = 0.$$

The sets of equations (14) and (15) when expressed in terms of the vectors $\vec{u}(u, v, w)$ and $\vec{\xi}(\xi, \eta, \zeta)$ take the forms

$$(\lambda+\mu)\vec{\nabla} e + \mu\nabla^2\vec{u} + A\vec{\nabla}\times\frac{\partial\vec{\xi}}{\partial t} = \varrho\frac{\partial^2\vec{u}}{\partial t^2} \tag{16}$$

$$-A\frac{\partial\vec{\xi}}{\partial t} + B\nabla^2(\vec{\xi}+\vec{\nabla}\times\vec{u}) = 0 \tag{17}$$

respectively, where

$$\vec{\nabla} \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z},\ \nabla^2 \equiv (\vec{\nabla}.\vec{\nabla}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

3. Love Waves. We consider a semi-infinite medium bounded by the plane $z=0$, and take the positive direction of the z-axis into the medium. A layer of thickness unity of different material is spread over the surface $z=0$ so that the upper surface of the layer is given by $z=-1$ (in dimensionless coordinates). Let $(\lambda, \mu, A, B, \varrho)$ and $(\bar{\lambda}, \bar{\mu}, \bar{A}, \bar{B}_1 \bar{\varrho})$ be the physical constants of the layer and the semi-infinite medium respectively. We consider the possibility of the propagation of a purely transverse wave (of Love type) in the medium such that the disturbance penetrates only a little distance into the interior. We assume the displacement and rotation components as

$$u=w=0,\ v=V(z)\exp\{i(kx-pt)\},\ i\equiv(-1)^{\frac{1}{2}} \tag{18}$$

$$(\xi, \eta, \zeta)=\{\xi_1(z), \eta_1(z)\zeta_1(z)\}\exp\{i(kx-pt)\} \tag{19}$$

where V, ξ_1, η_1, ζ_1 are functions of z only, p is a real positive constant and k is in general complex. The equations (16) now reduce to a single equation

$$\mu\nabla^2 v+A\frac{\partial}{\partial t}\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)=\varrho\frac{\partial^2 u}{\partial t^2}$$

which simplifies further to

$$\left(D^2+\frac{fp^2}{\mu}-k^2\right)V=\frac{Ap}{\mu}(iD\xi_1+k\zeta_1)\ ,\ D\equiv\frac{\partial}{\partial z} \tag{20}$$

while equations (17) imply

$$\begin{aligned}
&-A\frac{\partial\xi}{\partial t}+B\nabla^2\left(\xi-\frac{\partial v}{\partial z}\right)=0\\
&-A\frac{\partial\eta}{\partial t}+B\nabla^2\eta=0\\
&-A\frac{\partial\zeta}{\partial t}+B\nabla^2\left(\zeta+\frac{\partial v}{\partial x}\right)=0.
\end{aligned} \tag{21}$$

The first and third equations of (21) with the help of (18) and (19) reduce to

$$\begin{aligned}
&\left(D^2+\frac{iAp}{B}-k^2\right)\xi_1=D(D^2-k^2)V\\
&\left(D^2+\frac{iAp}{B}-k^2\right)\zeta_1=-ik(D-k^2)V
\end{aligned} \tag{22}$$

The elimination of ξ_1 and ζ_1 from (20) by the use of (22) gives

$$\left(D^2+\frac{\varrho p^2}{\mu}-k^2\right)\left(D^2+\frac{iAp}{B}-k^2\right)V=\frac{iAp}{\mu}(D^2-k^2)^2V \tag{23}$$

the solution of which is

$$V(z)=C_1e^{m_1z}+C_2e^{-m_1z}+D_1e^{m_2z}+D_2e^{-m_2z} \tag{24}$$

where C_1, C_2, D_1, D_2 are independent of z and $(\pm m_1, \pm m_2)$ are the roots of the equation

$$\left(m^2+\frac{\varrho p^2}{\mu}-k^2\right)\left(m^2+\frac{iAp}{B}-k^2\right) = \frac{iAp}{\mu}(m^2-k^2)^2. \tag{25}$$

If we write
$$h^2 = \frac{iAp}{B}-k^2 \tag{26}$$

the solutions of (22) are

$$\xi_1(z) = C_3e^{ihz}+C_4e^{-ihz}+\frac{D(D^2-k^2)}{D^2+h^2}V(z)$$

$$\zeta_1(z) = C_5e^{ihz}+C_6e^{-ihz}-\frac{ik(D^2-k^2)}{D^2+h^2}V(z).$$

If the value of $V(z)$ as given by (24) is substituted in these equations we have

$$\xi_1(z) = C_3e^{ihz}+C_4e^{-ihz}+m_1m_3^2(C_1e^{m_1z}-C_2e^{-m_1z})+m_2m^2_4(D_1e^{m_2z}-D_2e^{-m_2z}) \tag{27}$$

$$\zeta(z) = C_5e^{ihz}+C_6e^{-ihz}-ik\{m^2_3(C_1^{m_1z}+C_2e^{-m_1z})+m^2_4(D_1e^{m_2z}+D_2e^{-m_2z})\} \tag{28}$$

where
$$m^2_3 = (m^2_1-k^2)/(m_1^2+h^2),\ m^2_4 = (m_2^2-k^2)/(m_2^2+h^2).$$

If we substitute in (20) the values of V, ξ_1 and ζ_1 from (24), (27) and (28) it is found that the coefficients of $e^{\pm m_1z}$ and $e^{\pm m_2z}$ in the resulting equation are identically zero by virtue of the equation (25), and we obtain the reduced equation

$$-h(C_3e^{ihz}-C_4e^{-ihz})+k(C_5e^{ihz}+C_6e^{-ihz}) = 0.$$

Since the above relation holds for all values of z, the coefficients of e^{ihz} and e^{-ihz} must vanish separately and hence

$$C_5 = (h/k)C_3,\ C_6 = -(h/k)C_4, \tag{29}$$

The assumptions (18) and (19) imply that

$$\sigma_{zx} = \sigma_{zz} = 0, \quad \sigma_{zy} = \mu e^{i(kx-pt)}DV \tag{30}$$

$$\sigma'_{zx} = ipA\eta_1e^{i(kx-pt)}, \quad \sigma'_{zy} = -ipA\xi_1e^{i(kx-pt)} \tag{31}$$

$$\left.\begin{aligned} M_{zy} &= Be^{i(kx-pt)}D\eta_1 \\ M_{zx} &= Be^{i(kx-pt)}D(\xi_1-DV) \\ M_{zz} &= Be^{i(kx-pt)}D(\zeta_1+ikV) \end{aligned}\right\} \tag{32}$$

In the subsequent discussion a quantity (except z, k and p) with a bar over it will denote its value in the semi-infinite medium $z > 0$ while a quantity without a bar will denote its value in the layer $-1 < z < 0$. Guided by the relations (24), (27), (28) and (29) we write, for the lower mass $z > 0$, the relations

$$\bar{V}(z) = \bar{C}_2 e^{-\bar{m}_1 z} + \bar{D}_2 e^{-\bar{m}_2 z} \tag{33}$$

$$\bar{\xi}_1(z) = \bar{C}_4 e^{-i\bar{h}z} - \bar{m}_1 \bar{m}_3^{\,2} \bar{C}_2 e^{-\bar{m}_1 z} - \bar{m}_2 \bar{m}_4^{\,2} \bar{D}_2 e^{-\bar{m}_2 z} \tag{34}$$

$$\bar{\zeta}_1(z) = \bar{C}_6 e^{-i\bar{h}z} - ik\{\bar{m}_3^{\,2} \bar{C}_2 e^{-\bar{m}_1 z} + \bar{m}_4^{\,2} \bar{D}_2 e^{-\bar{m}_2 z}\} \tag{35}$$

with

$$\bar{C}_6 = -(\bar{h}/k)\bar{C}_4 \tag{36}$$

If the real parts of $\bar{m}_1$ and $\bar{m}_2$ are positive while the imaginary part of $\bar{h}$ is negative each of $\bar{V}(z)$, $\bar{\xi}(z)$ and $\bar{\zeta}(z)$ vanishes as $z \to \infty$.

4. Boundary Conditions. The value of η as assumed in (19) will satisfy the second equation of (21), the continuity conditions

$$\eta = \bar{\eta} \quad \sigma'_{zx} = \bar{\sigma}'_{zx}, \; M_{zy} = \bar{M}_{zy} \qquad \text{on} \quad z = 0,$$

and the free surface conditions

$$\sigma'_{zx} = 0, \quad M_{zy} = 0 \qquad \text{on} \quad z = -1$$

provided $\eta = 0$, as is evident from the first equations of (31) and (32). The continuity conditions

$$V(z) = \bar{V}(z), \; \xi(z) = \bar{\xi}(z), \; \zeta(z) = \bar{\zeta}(z) \qquad \text{on} \quad z = 0$$

give respectively

$$C_1 + C_2 + D_1 + D_2 = \bar{C}_2 + \bar{D}_2 \tag{37}$$

$$C_3 + C_4 + m_1 m_3^{\,2}(C_1 - C_2) + m_2 m_4^{\,2}(D_1 - D_2) = \bar{C}_4 - \bar{m}_1 \bar{m}_3^{\,2} \bar{C}_2 - \bar{m}_2 \bar{m}_4^{\,2} \bar{D}_2 \tag{38}$$

$$C_5 + C_6 - ik\{m_3^{\,2}(C_1 + C_2) + m_4^{\,2}(D_1 + D_2)\} = \bar{C}_6 - ik(\bar{m}_3^{\,2} \bar{C}_2 + \bar{m}_4^{\,2} \bar{D}_2) \tag{39}$$

which follow from equations (24), (27), (28), (33), (34) and (35). It is seen from the last equations of (30) and (31) that the continuity condition $\tau_{zy} = \bar{\tau}_{zy}$ on $z = 0$ implies

$$\mu DV - ipA\xi_1 = \bar{\mu}\bar{D}\bar{V} - ip\bar{A}\bar{\xi} \qquad \text{on} \quad z = 0$$

which yields

$$\left(1 - \frac{iAp}{\mu} m_3^{\,2}\right) m_1 (C_1 - C_2) + \left(1 - \frac{iAp}{\mu} m_4^{\,2}\right) m_2 (D_1 - D_2) - \frac{iAp}{\mu}(C_3 + C_4)$$

$$+ \frac{\bar{\mu}}{\mu}\left[\left(1 - \frac{i\bar{A}p}{\bar{\mu}} \bar{m}_3^{\,2}\right) \bar{m}_1 \bar{C}_2 + \left(1 - \frac{i\bar{A}p}{\bar{\mu}} \bar{m}_4^{\,2}\right) \bar{m}_2 \bar{D}_2 + \frac{i\bar{A}p}{\bar{\mu}} \bar{C}_4\right] = 0. \tag{40}$$

Again, utilizing the last two relations of (32) and the continuity conditions

$$M_{zx} = \bar{M}_{zx},\ M_{zz} = \bar{M}_{zz} \qquad \text{on} \quad z = 0,$$

we have

$$Bh(C_3 - C_4) - Ap\{m_5^2(C_1 + C_2) + m_6^2(D_1 + D_2)\} + \bar{B}\bar{h}\bar{C}_4 + \bar{A}p(\bar{m}_5^2\bar{C}_2 + \bar{m}_6^2\bar{D}_2) = 0 \tag{41}$$

$$iBh(C_5 - C_6) - Apk\{m_5^2(C_1 - C_2) + m_6^2(D_1 - D_2)\} + i\bar{B}\bar{h}\bar{C}_6 - \bar{A}pk(\bar{m}_5^2\bar{C}_2 + \bar{m}_6^2\bar{D}_2) = 0 \tag{42}$$

where $\quad m_5^2 = m_1^2/(m_1^2 + h^2),\ m_6^2 = m_2^2/(m_2^2 + h^2).$

The boundary conditions

$$\tau_{zy} = M_{zx} = M_{zz} = 0 \qquad \text{on} \quad z = -1$$

which are equivalent to

$$\mu DV - ipA\xi_1 = D(\xi_1 - DV) = D(\zeta_1 + ikV) = 0 \quad \text{on } z = -1$$

as seen from the last equations of (30) and (31) and the last two equations of (32). These three conditions give

$$\left(1 - \frac{iAp}{\mu} m_3^2\right) m_1(C_1 e^{-m_1} - C_2 e^{m_1}) + \left(1 - \frac{iAp}{\mu} m_4^2\right) m_2(D_1 e^{-m_2} - D_2 e^{m_2}) - \frac{iAp}{\mu}(C_3 e^{-ih} + C_4 e^{ih}) = 0 \tag{43}$$

$$h(C_3 e^{-ih} - C_4 e^{ih}) - \frac{Ap}{B}\{m_5^2(C_1 e^{-m_1} + C_2 e^{m_1}) + m_6^2(D_1 e^{-m_2} + D_2 e^{m_2})\} = 0 \tag{44}$$

$$h(C_5 e^{-ih} - C_6 e^{ih}) - \frac{Apk}{B}\{m_5^2(C_1 e^{-m_1} - C_2 e^{m_1}) + m_6^2(D_1 e^{-m_2} - D_2 e^{m_2})\} = 0. \tag{45}$$

5. Wave-velocity Equation. If we put

$$E_1 = C_1 + C_2, \quad E_2 = C_1 - C_2, \quad E_3 = D_1 + D_2, \quad E_4 = D_1 - D_2$$
$$E_5 = C_3 + C_4, \quad E_6 = C_3 - C_4$$

so that

$$2C_1 = E_1 + E_2, \quad 2C_2 = E_1 - E_2, \quad 2D_1 = E_3 + E_4, \quad 2D_2 = E_3 - E_4$$
$$2C_3 = E_5 + E_6, \quad 2C_4 = E_5 - E_6$$

and use the relations (29) and (36), then the equations from (37) to (45) take the forms

$$E_1 + E_3 = \bar{C}_2 + \bar{D}_2 \tag{37a}$$

$$E_5 + m_1 m_3^2 E_2 + m_2 m_4^2 E_4 = \bar{C}_4 - \bar{m}_1 \bar{m}_3^2 \bar{C}_2 - \bar{m}_2 \bar{m}_4^2 \bar{D}_2 \tag{38a}$$

$$hE_6 - ik^2(m_3^2 E_1 + m_4^2 E_3) + \bar{h}\bar{C}_4 + ik^2(\bar{m}_3^2 \bar{C}_2 + \bar{m}_4^2 \bar{D}_2) = 0 \tag{39a}$$

$$m_1 m_7 E_2 + m_2 m_8 E_4 - \frac{iAp}{\mu} E_5 + \frac{\bar{\mu}}{\mu}\left(\bar{m}_1 \bar{m}_7 \bar{C}_2 + \bar{m}_2 \bar{m}_8 \bar{D}_2 + \frac{i\bar{A}p}{\mu} \bar{C}_4\right) = 0 \tag{40a}$$

$$BhE_6-Ap(m_5^2E_1+m_6^2E_3)+\bar{B}\bar{h}\bar{C}_4+\bar{A}p(\bar{m}_5^2C_2+\bar{m}_6^2\bar{D}_2)=0 \qquad (41a)$$

$$iBh^2E_5-Apk^2(m_5^2E_2+m_6^2E_4)-i\bar{B}\bar{h}^2\bar{C}_1-\bar{A}pk^2(\bar{m}_5^2\bar{C}_2+\bar{m}_6^2\bar{D}_2)=0 \qquad (42a)$$

$$m_1m_7(E_2\cosh m_1-E_1\sinh m_1)+m_2m_8(E_4\cosh m_2-E_3\sinh m_2)$$
$$-\frac{iAp}{\mu}(E_5\cos h-iE_6\sin h)=0 \qquad (43a)$$

$$h(E_6\cos h-iE_5\sin h)-\frac{Ap}{B}\{m_5^2(E_1\cosh m_1-E_2\sinh m_1)$$
$$+m_6^2(E_3\cosh m_2-E_4\sinh m_2)\}=0 \qquad (44a)$$

$$h^2(E_5\cos h-iE_6\sin h)-\frac{Apk^2}{B}\{m_5^2(E_2\cosh m_1-E_1\sinh m_1)$$
$$+m_6^2(E_4\cosh m_2-E_3\sinh m_2)\}=0 \qquad (45a)$$

where
$$m_7=1-\frac{iAp}{\mu}m_3^2,\; m_8=1-\frac{iAp}{\mu}m_4^2. \qquad (46)$$

The above nine relations from (37a) to (45a) may be considered as the linear homogeneous simultaneous equations in the nine variables $E_1, E_2, \ldots E_6, \bar{C}_2, \bar{D}_2$ and $\bar{C}_4$ and consequently these variables can be eliminated determinantally to give an equation in k anp p which is the wave-velocity equation. Now in order to obtain this equation we arrange the different terms of the equations from (37a) to (45a) in such a way that they involve the unknown coefficients $E_1, E_2, \ldots E_6, \bar{C}_2, \bar{D}_2, \bar{C}_4$ in the order as written. When these are eliminated we get the wave velocity equation involving a determinant of ninth order as follows.

$$\begin{vmatrix}
1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & m_1m_3^2 & 0 & m_2m_4^2 & 1 & 0 & \bar{m}_1\bar{m}_3^2 & \bar{m}^2\bar{m}_4^2 & -1 \\
-im_3^2k^2 & 0 & -im_4^2k^2 & 0 & 0 & h & i\bar{m}_3^2k^2 & i\bar{m}_4^2k^2 & \bar{h} \\
0 & m_1m_7 & 0 & m_1m_8 & -\dfrac{iAp}{\mu} & 0 & \dfrac{\bar{\mu}}{\mu}\bar{m}_1\bar{m}_7 & \dfrac{\bar{\mu}}{\mu}\bar{m}_2\bar{m}_8 & \dfrac{i\bar{A}p}{\mu} \\
-Am_5^2p & 0 & -Am_6^2p & 0 & 0 & Bh & A\bar{m}_5^2p & Apm_6^2 & \bar{B}\bar{h} \\
0 & -Am_5^2pk^2 & 0 & -Am_6^2pk^2 & iBh^3 & 0 & -\bar{A}\bar{m}_5^2pk^2 & -\bar{A}\bar{m}_6^2pk^2 & i\bar{B}\bar{h}^2 \\
\begin{matrix}-m_1m_7\times\\ \sinh m_1\end{matrix} & \begin{matrix}m_1m_7\times\\ \cosh m_1\end{matrix} & \begin{matrix}-m_2m_8\times\\ \sinh m_2\end{matrix} & \begin{matrix}m_2m_8\times\\ \cosh m_2\end{matrix} & -\dfrac{Ap}{\mu}\cos h & -\dfrac{Ap}{\mu}\sin h & 0 & 0 & 0 \\
\begin{matrix}-Am_5^2p\times\\ \cosh m_1\end{matrix} & \begin{matrix}Am_5^2p\times\\ \sinh m_1\end{matrix} & \begin{matrix}-Am_6^2p\times\\ \cosh m_2\end{matrix} & \begin{matrix}Am_6^2p\times\\ \sinh m_2\end{matrix} & \begin{matrix}-ihB\times\\ \sin h\end{matrix} & \begin{matrix}hB\times\\ \cos h\end{matrix} & 0 & 0 & 0 \\
\begin{matrix}Am_5^2pk^2\times\\ \sinh m_1\end{matrix} & \begin{matrix}-Am_5^2pk^2\times\\ \cosh m_1\end{matrix} & \begin{matrix}Am_6^2pk\times\\ \sinh m_2\end{matrix} & \begin{matrix}-Am_6^2pk^2\times\\ \cosh m_2\end{matrix} & \begin{matrix}h^2B\times\\ \cos h\end{matrix} & \begin{matrix}-ih^2B\times\\ \sin h\end{matrix} & 0 & 0 & 0
\end{vmatrix}=0.$$

As p is assumed to be a known real positive quantity this equation determines k in terms of p. Since, k is in general a complex number its imaginary part contributes to the attenuation of waves. The numerical solution of this equation will be communicated n a subsequent publication.

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BOOK REVIEW

Shaukat Abbas—*Elements of Calculus*, Educational Book Depot, Hyderabad (W. Pak.), 1958, xiv + 304 pp. Rs. 5/8.

This book on Differential and Integral Calculus is meant for students the Intermediate Classes of Pakistani Universities and as such it does not contain detailed discussion of the topics involved, though rigour has not always been sacrificed. The author seems to be quite aware of the limitations of such a book and he has been partly successful in presenting the subject matter in such a manner as can be understood by students of the Intermediate Classes. However the reviewer believes that the author should have illustrated the ideas of limit and continuity by means a fair number of graphs as the book is meant for beginners. The number of illustrative examples worked out is fairly large.

The quality of printing is not good.

S. K. DATTA

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